CONTRIBUTIONS TO FIXED POINT THEORY IN CONE METRIC SPACES

FINAL PROJECT REPORT

Submitted for Financial Assistance

Under

UGC Minor Research Project For Teachers Under XII Plan (2012-2017)

Submitted To UGC - SERO

BAHADUR SHAH ZAFAR MARG, NEW DELHI – 110 002

in

MATHEMATICS

By

Principal Investigator

Capt. KAMBHAMPATI SUJATHA, M.Sc., M.phil.

No. F MRP – 4536/14 (SERO/UGC), Proposal No. 587 (2014-2016)

Lecturer in Mathematics, St. Joseph's College for Women (A)

Visakhapatnam, 530004

UGC

University Grants Commission

2016

CONTENTS

Acknowledgements

It has been a great privilege for me to complete the Minor Research Project in Mathematics with the financial help of UGC link no. 4536/14 UGC – SERO. I sincerely thank Deputy Secretary, UGC – SERO, Hyderabad, for granting me the MRP.

I deem it a privilege to extend my deepest and heartfelt gratitude to **Sr. Alice Marie, Correspondent** and **Dr. Sr. P. D. Shyji**, **Principal St. Joseph's College For Women (Autonomous)**, Visakhapatnam for readily granting permission to undertake this UGC Minor Research Project activity and for their support and encouragement from time to time.

I am also thankful to **Dr. Sr. N.D .Veronica**, Former Principal and **Sr. K.C.Mary, Office Superintendent** of the College for all their cooperation in completing this Project.

I offer my sincere thanks to **Dr. K.P.R. Sastry**, Retired Professor, Department of Mathematics, Andhra University, Visakhapatnam for sparing his valuable time; without his cooperation, it might not have been possible to complete the Project.

I am also thankful to **Mrs. P. Krishnakumari** , Head , Department of Mathematics, **St. Joseph's College For Women (Autonomous)**, Visakhapatnam without whose interest, it might not have been possible to complete the Project.

I am very thankful to all of my **colleagues, especially Dr. P.Bhushanavathi and Dr.P.Sarada** for their help and encouragement throughout the research work.

I am thankful to Dr. Zhang Xian, Dr. Sh. Rezapour, Dr. G. Jungck, Dr. I. Sahin and Dr. N. Metiya for their immediate response to send their published papers to me.

None of my efforts would have been fruitful but for the constant encouragement and ever inspiring attitude of my husband **Sri. M.Venkata Ravi Chandra.** I shall be failing in my duty if I forget to acknowledge our son **M.V. Sai Sri Harsha** to complete my UGC Minor Research Project without any distraction.

Thank you all

Kambhampati Sujatha

Chapter 0

Introduction to Fixed Point Theory

0.1 Introduction

Definitions 0.1:

Suppose X is a non empty set. A mapping $T: X \rightarrow X$ is called a self map of X. If there is an element $x \in X$ such that $Tx = x$, then x is called a fixed point of the self map T of X.

Definition 0.2 :

A topological space X is said to be a fixed point space if every continuous map of X into itself has a fixed point.

Definition 0.3 :

Let (X,d) be a complete metric space and $T : X \rightarrow X$. Then T is said to be contraction mapping if for all $x , y \in X$, $d(Tx , Ty) \leq k\, d (x , y)$ … (0.3.1)

where $0 < k < 1$.

It is easy to see that the contraction mapping principle, any mapping T satisfying (0.3.1) will have a unique fixed point.

Definition 0.4 :

A metric on a non empty set X is a function (called the distance function or simply distance) $d : X \times X \rightarrow R$ (where R is the set of real numbers). For all *x*, *y*, *z* in X, this function is required to satisfy the following conditions:

1. $d(x, y) \geq 0$ (non-negativity)
2. $d(x, y) = 0$ if and only if $x = y$
3. $d(x, y) = d(y, x)$ (symmetry)
4. $d(x, z) \leq d(x, y) + d(y, z)$ (subadditivity / triangle inequality).

The first condition is implied by the others.

Definition 0.5 : (Huang , Xian ,[14]) Let E be a real Banach space and P a subset of E. P is called a cone if

(i) P is closed, non-empty and $P \neq \{ 0 \}$

(ii) $ax + by \in P \;\; \forall \; x, y \in P$ and non-negative real numbers a and b.

(iii) $P \cap (-P) = \{0\}$.

Definition 0.6 : (Huang , Xian ,[14])

E is a real Banach space, P is a cone in E with int $P \neq \emptyset$ and $\leq$ is the partial ordering with respect to P. Let X be a non-empty set and $d: X \times X \to P$ a mapping such that.

(d_1) $0 \leq d(x,y)$ for all $x, y \in X$ (non - negativity)

(d_2) $d(x,y) = 0$ if and only if $x = y$.

(d_3) $d(x,y) = d(y,x)$ for all $x, y \in X$ (symmetry)

(d_4) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$ (triangle inequality)

Then d is called a cone-metric on X and (X, d) is called a cone metric space.

Definition 0.9 (Lakzian, Arabyani [19] , Definition 2.4) :

Let (X,d) be a cone metric space . Then a mapping $p : X \times X \to E$ is called *w* - distance on X, if the following are satisfied :

(a) $0 \leq p(x, y)$ for all $x, y \in X$

(b) $p(x, z) \leq p(x, y) + p(y, z)$ for all $x, y, z \in X$

(c) $p(x, .) \to E$ is lower semi-continuous for all $x \in X$.

(d) For any $0 << \alpha$, there exists $0 << \beta$ such that $p(z, x) << \beta$ and $p(z, y) << \beta$ imply $d(x, y) << \alpha$ where $\alpha, \beta \in E$.

Fixed point theorems provide conditions under which maps (single or multivalued) have solutions. The theory itself is a beautiful mixture of analysis, topology, and geometry. Over the last 100 years or so the theory of fixed points has been revealed as a very powerful and important tool in the study of nonlinear phenomena. In particular fixed point techniques have been applied in such diverse fields as biology, chemistry, economics, engineering, game theory and physics. Fixed point theory plays an important role in functional analysis, approximation theory, differential equations and applications such as boundary value problems etc.(Definition 0.1)

Fixed point theory broadly speaking demonstrates the existence, uniqueness and construction of fixed points of a function or a family of functions under diverse assumptions about the structure of the domain X (such as X may be a metric space or normed linear space or a topological space) of the concerned functions.

The methods of the theory vary over almost all mathematical techniques. There are many works entirely devoted to fixed point theory such as [33],[35],[7]

Brouwer [7] published his fixed point theorem in 1911 for finite dimensional spaces, now known as Brouwer fixed point theorem. Brouwer was elected to the Royal Academy of Sciences in 1912 and in the same year, was appointed extraordinary professor of set theory, function theory and axiomatic at University of Amsterdam; held the post until he retired in 1951
(Definition 0.2).

In 1910, Brouwer proved that the closed unit ball of R^n is a fixed point space. Perhaps the most important result in fixed point theory is the famous theorem of Brouwer.

In 1930, Schauder [32] extended the result to arbitrary Banach spaces (i.e. to Banach Spaces which do not necessarily admit a basis) . In the context of Banach and Schauder's theorems, they have led to a number of interesting results as well as new techniques for further application.

Banach founded modern functional analysis and made major contributions to the theory of topological vector spaces. In addition , he contributed to measure theory, integration, the theory of sets and orthogonal series. In his dissertation, written in 1920, he defined axiomatically what today is called a Banach space. Banach's fixed point theorem has many applications in solving non-linear equations, but suffers from one drawback that the contraction condition forces T to be continuous on X.

The famous Banach Contraction Principle states that every contraction in a complete metric space has a unique fixed point. It has two core hypotheses: completeness and contractivity. Both notions depend on the definition of the underlying metric. Much recent work has focused on the extension of the notion of metric spaces and the related notion of contractivity.

In past few decades study of fixed point theory is one of the most interesting fields to researchers. In this direction Banach contraction mapping principle is one of the most interesting result (Definition 0.3)

The concept of a metric space was introduced in 1906 by Frechet [13] which furnishes the common idealization of a large number of mathematical, physical and other scientific constructs in which the notion of a "distance" appears. The objects under consideration may be most varied. They may be points, functions, sets and even the subjective experiences of sensations. What matters is the possibility of associating a non negative real number with each ordered pair of elements of certain set, and that the number associated with pairs and triples of such elements satisfies certain conditions.

A metric or distance function is a function which defines a distance between elements of a set. A set with a metric is called a metric space. A metric induces a topology on a set but not all topologies can be generated by a metric. When a topological space has a topology that can be described by a metric, we say that the topological space is metrizable (Definition 0.4).

Fixed Point Theory in Cone Metric Spaces : One extension of metric spaces is the so called cone metric space. In cone metric spaces, the metric is no longer a positive number but a vector, in general an element of a Banach space equipped with a cone.

In 1906, the French mathematician Maurice Frechet [8, 13] introduced the concept of metric spaces, although the name “metric” is due to Hausdorff [8, 12]. In 1934, Duro Kurepa[18], proposed metric spaces in which an ordered vector space is used as the codomain of a metric instead of the set of real numbers. In literature Metric Spaces with Vector Valued Metrics are known under various names such as Pseudo Metric Spaces , K-Metric Spaces, Generalized Metric Spaces, Cone-Valued Metric Spaces, Cone Metric Spaces , Abstract Metric Spaces and Vector Valued Metric Spaces. Fixed point theory in K-metric spaces was developed by Perov in 1964 [28, 29].

In 2007 , Huang , Xian [14] suggested the notion of a cone metric space and established some fixed point theorems in cone metric spaces, an ambient space which is obtained by replacing the real axis in the definition of the distance, by an ordered real Banach space whose order is induced by a normal cone P (Definition 0.5 & 0.6)

Huang , Xian [14] proved some fixed point theorems of contractive mappings , which generalize the existing results in metric spaces such as Banach [5] , Kannan [16] etc.

In 2008, Rezapour and Hamlbarani [31] , proved that there are no normal cones with normal constant $M < 1$. Further , in [31] it was shown that for $k > 1$ there are cones with normal constant $M > k$. An example of a non normal cone is given in [31]. Further , Rezapour and Hamlbarani [31] obtained generalizations of the results of Huang , Xian [14] (Theorems 1.19,1.21 and 1.22) by removing the assumption of normality of the cone.

In 2008, Abbas and Jungck [1] derived several coincidence and common fixed point theorems for mappings defined on a cone metric space.

In 2009, Arshad, Azam and Vetro [25] established some results on points of coincidence and common fixed points for three self mappings and these observations generalized the results of Abbas and Jungck [1] (Theorem 1.37)

Azam and Arshad [4] have further improved theorems of Abbas and Jungck [1] (Theorem 1.37) and Huang , Xian [14] (Theorems 1.19, 1.21 , 1.22 , 1.23)

In 2010 , Berinde [36] , derived coincidence and common fixed point theorems, similar to those in Abbas and Jungck [1] (Theorem 1.37) , but for a more general class of almost contractions.

In 2011 , Olaleru [15] , extended the results of Arshad, Azam and Vetro [23] (Lemma1.43,Proposition1.44 ,Theorem 1.45 , Theorem 1.46)

In 2011, Mehta and Joshi [20] , generalized the result of Abbas and Jungck [2] (Theorem 1.37)

In 2010 , Abbas , Rhoades , Nazir [26] , extended the results of Berinde [36] (Theorem 1.55) . They have proved the existence of coincidence points and common fixed points for four mappings satisfying generalized contractive conditions without exploiting the notion of continuity of any map involved therein, in a cone metric space.

In 2010, Amit Singh , Dimri and Bhatt [3] proved a unique common fixed point theorem for four maps using the notion of weak compatibility without using the notion of continuity which generalized and extended the results of Abbas and Rhoades [2].

Fixed Point Theory of Multifuctions in Cone Metric Spaces :

In 2010 , Dimri , Amit Singh and Bhatt [11] , proved common fixed point theorems for two multivalued maps in cone metric spaces with normal constant M = 1 which generalized and extended the results of Rezapour [30] .

In 2011 , Dhanorkar and Salunke [9] generalized the results of Rezapour [30] about common fixed points of two multifunctions on cone metric spaces with normal constant M = 1.

Fixed Point Theory of T – Contractive Mappings in Cone Metric Spaces :

In 2008, Morales and Rojas [23] have extended the definition of a T-contractive map to cone metric space. Morales and Rojas [23] extended the result of Guang and Zhang [14] (Theorem 1.19) to T- contraction maps.

In 2009, Moradi [22] introduced the notion of T-Kannan contractive mappings in metric spaces which extends the notion of Kannan type contractions [16].

In 2009, Morales and Rojas [24] analyzed the existence of fixed points of T-Kannan type contractive mappings defined on a complete cone metric space (M, d). Morales and Rojas [24] proposed the notion of T-Chatterjea Mapping. Morales and E.Rojas [24] , extended the result (Theorem 1.22) of Guang and Zhang [14] and Moradi [22] (Theorem 2.1) to T- Kannan type contraction (T K_1− Contraction). Morales and Rojas [24] , extended the result (Theorem1.23) of Guang and Zhang [14] to T- Kannan type contraction (T K_2 – Contraction). In 1968 , Kannan [16] established a fixed point theorem. In 1968 , Kannan [16] established a fixed point theorem. It is interesting that Kannan's theorem is independent of the Banach contraction principle . Kannan's fixed point theorem is very important because Subrahmanyam [34] proved that Kannan's theorem characterizes the metric completeness. That is, a Metric space X is complete if and only if every Kannan mapping on X has a fixed point.

In 2009 , Branciari [6] , introduced the notion of cone rectangular metric spaces by replacing the triangular inequality of a cone metric space by a rectangular inequality and they investigated some common fixed point theorems for different types of contractive mappings in cone metric spaces. It is to be noted that any cone metric space is a cone rectangular metric space but the converse is not true in general. In 2009 , Jleli and Samet [21] extended Kannan's fixed point theorem in a rectangular metric space.

Fixed Point Theory involving *w*-distance and c-distance

The notion of a metric space with *w* - distance was introduced in 1996 by Osama Kada, Tomonari Suzuki and Wataru Takahashi [27].

In 2009, H. Lakzian, F.Arabyani [17] extended the above notion and introduced the notion of cone metric spaces with *w* -distance and proved some fixed point theorems (Definition 0.9)

In 2011 , G. A. Dhanorkar and J. N. Salunke [10] , continued the study of fixed points for self maps on cone metric spaces with *w*- distance.

References

1. **Abbas, M. and Jungck, G.** "Common fixed point results for non commuting mappings without continuity in cone metric spaces", J.Math.Anal.Appl.341,(2008), 416-420.
2. **Abbas, M. and Rhoades, B. E.** "Fixed and periodic results in cone metric spaces" Appl.Math.Lett, (2008), 22 (4), 511-515
3. **Amit Singh , Dimri, R. C. and Sandeep Bhatt.** "A Unique Common Fixed Point Theorem for Four Maps in Cone Matric Spaces", Int. Journal of Math. Analysis, Vol. 4, (2010), No. 31, 1511-1517.
4. **Azam, A. and Arshad, M**. "Common fixed points of generalized contractive maps in cone metric spaces;" Bulleton of the Iranian Mathematical Society Vol.35, (2009), No.2, 255-264.

5. **Banach, S. "**Surles operations dans les assembles abstract at leur applications aux equations integrables", Fund. Math. Soc. 3, (1922), 131-181.
6. **Branciari, A.** "A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces", Publ. Math. Debrecen, 57 (1-2),(2000), 31{37. 1, 1.1, 1}
7. **Brouwer, L.E.J. "**Uber eineindeutige, stetiger Transformationen" , von Flaachen in sich. Math. Ann. 69, (1910), 176-180.
8. **Deza, M. and Deza, E.** "Encyclopedia of Distances" , Springer-Verlag, (2009), i – x.
9. **Dhanorkar, G.A. and Salunke, J.N. "**A Generalization Common Fixed Points of Multifunctions on Cone Matrix Spaces,Applied Mathematical Sciences",Vol.5, (2011), no. 45, 2241 – 2247.
10. **Dhanorkar, G. A. and Salunke, J. N. "**A Generalization on Fixed Point Theorem on Cone Metric Spaces with w-Distance**"** , International Mathematical Forum, Vol. 6, (2011), no. 39 , 1915 - 1919
11. **Dimri, R.C. , Amit Singh and Sandeep Bhatt.** "Common Fixed Point Theorems for Multivalued Maps in Cone Metric Spaces" , Int. Math. Forum,Vol. 5 , (2010), no. 46 , pp. 2271-2278.
12. **Frechet, M. "** Sur quelques points du clacul fonctionnel, Rendiconti decircolo Matematic O di Palermo**"**, 22, (1906), 1-76.
13. **Hausdorff, F. "**Grundzüge der Mengenlehre, Verlag Von Veit & Company**"**, Leipzig (1914). Reprinted by Chelsea Publishing Company, New York (1949).
14. **Huang Long – Guang , Zhan Xian. "**Cone metric spaces and fixed point theorems of contractive mapping**"** J.Math.Anal.Appl. 332, (2007), 1468 - 1476.
15. **Johnson Olaleru. "**Common Fixed Points of Three Self – Mappings in Cone Metric Spaces" , Applied Mathematics E - Notes , 11, (2011) ,41-49
16. **Kannan, R. "**Some results on fixed points**"** , Bull. Calcutta Math. Soc.,60, (1968) , 71-76.
17. **Krasnoselskii, M.A. "**Topological Methods in the theory of nonlinear Integral equations**" ,** Pergamon press, Oxford (1964).
18. **Kurepa, D.R. "** Tableaux ramifies d'ensembles.Espaces pseudo-distancies**"**, C. R. Acad. Sci. Paris, 198 ,(1934), 1563–1565.
19. **Lakzian, H. and Arabyani, F. "**Some fixed point theorems in cone metric spaces with w-distance**"** , J.Math.Anay.Vol.3,(2009) ,no.22, 1081-1086.

20. **Mehta, J.G. and Joshi, M.L.** **“**On Complete Cone Metric Space and Fixed Point Theorem**”** , J. Sci. Res. 3(2) ,(2011), pp. 303-309.

21. **Mohamed Jleli and Bessem Samet.** **“**The Kannan’s Fixed Point Theorem in a Cone Rectangular Metric Space**”**, J.Nonlinear Sci. Appl. 2, (2009), no.3, 161-167.

22. **Moradi, S.** **“**Kannan fixed point theorem on complete metric spaces and on generalized metric spaces depend on another function**”** arXiv:0903.1577V1 [Mat.FA].

23. **Morales, J .R. and Edixon Rojas**. **“**Cone metric spaces and fixed point theorems of T-Contractive mapping**”** , n Revista notas de metrmatica, Vol.4(2),(2008), no.269,66-78.

24. **Morales Jose, R. and Rojas, E.** **“**Cone metric space and fixed point theorem of T-Kannan contractive mapping**”** , arXiV : 0907.3949v1.

25. **Muhammad Arshad , Akbar Azam and Pasquale Vetro.** Research Article, **“**Some Common Fixed Point Results in Cone Metric Spaces**”**, Hindawi Publishing Corporation , Fixed Point Theory and Application , Volume 1 ,(2009), Article ID 493965, 11 pages.

26. **Mujahid Abbas , Rhodes, B.E. , Talat Nazir. “** Common Fixed Points for Four Maps in Cone Metric Spaces**”** , Appl. Math. Comput. , (2010).

27. **Osama Kada, Tomonari Suzuki and Wataru Takahashi.** **“**Non convex minimization theorems and fixed point theorems in complete metric spaces**”** , Math.Japonica,44,(1996),(381-591).

28. **Perov, A.I.** **“**On Cauchy problem for a system of ordinary differential equations”,Pviblizhen. Met. Reshen. Differ. Uvavn., 2,(1964),115-134.

29. **Proinov, P.D.** **“**A unified theory of cone metric spaces and its applications to the fixed point**”** In press, http://arxiv.org/abs/1111.4920.

30. **Rezapour, S.** **“** Two results about common fixed point Multifunctions on cone metric spaces **”,** Mathematical Proceedings of the Royal Irish Academy 109, (2009) ,155 - 161; doi:10.3318/PRIA.2009.109.2.155.

31. **Rezapour, Sh. and Hamlbarani, R.** **“**Some notes on the paper cone metric spaces and fixed point theorems of contractive mappings**”**, J.Math, Anal. Appl., 345, (2008), 719 – 724.

32. **Schauder, J.** **“**Zur Theoric stetiger Abbildungen in Funktionalrälumen**”,** Math. Z. 26, (1927), 47-65, 417-431.

33. **Smart, D.R.** **“** Fixed point theorems **”** , Cambridge tract in Mathematics, (1974) , Cambridge University press , no.66
34. **Subrahmanyam, P.V.** **“** Completeness and Fixed Points**”** , Monatshefte Fiir Mathematik , Vol.80 ,(1975), no.4 , pp. 325- 330 .
35. **Vanderwalt, T.** **“** Fixed and almost fixed points” , Mathematical Centre tracts 1, Amsterdam (1967).
36. **Vasile Berinde.** **“** Common Fixed Points of Non-Commuting Almost Contractions in Cone Metric Spaces**”**, Math.Commun. , Vol.15, (2010) , no. 1,pp. 229-241.

CHAPTER 1

Common Fixed Point Theorem for Two Self Maps in a Cone Metric Space with ω – Distance

1 Introduction

Haung and Zhang [2] generalized the concept of metric Space, replacing the set of real numbers by an ordered Banach space, and obtained some fixed point theorems for mappings satisfying different contractive conditions. The metric space with *w*-distance was introduced by O. Kada et al [5]. These two concepts were combined together and Cone metric space with *w*− distance was introduced by H.Lakzian and F.Arabyani [6], and a fixed point theorem was proved. Abbas and Jungck [1] proved some common fixed point theorems for weakly compatible mappings in the setting of cone metric space. D. IIić and Rakoćević [3], Rezapour and Hamlbarani [7] also proved some common fixed point theorems on cone metric spaces. Our objective is to extend these concepts together to establish a common fixed point theorem for a pair of weakly compatible mappings in cone metric space using *w*-distance. Consequently, we improve and generalize various results existing in the literature.

In this paper , we extend results of Sami Ullah Khan and Arjamand Bano [8] and prove some common fixed point theorem for a pair of weakly compatible mappings in cone metric spaces using ω – distance on X without using normality in cone metric space.

2 Preliminaries

2.1 Definition: (L.G. Haung and X. Zhang [2])

Let E be a real Branch Space and P a subset of E. The set P is called a cone if

1. P is closed, non-empty and $P \neq \{0\}$;

2. $a,b \in R$, a, b ≥ 0, $x, y \in P$ then $ax + by \in P$;

3. P ∩ (-P) = {0}

For a given cone P of E ,we define a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in$ P. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x << y$ stands for $y - x \in$ Int P, where Int P denotes the interior of P.

2.2 Definition: (L.G. Haung and X. Zhang [2])

Let E be a real Banach space and P be a cone of E . The cone P is called normal if there is a number K > 0 such that for all $x, y \in$ E, $0 \leq x \leq y$ implies $\| x \| \leq K \| y \|$

The least positive number K satisfying the above inequality is called the normal constant of P. In the following, we always suppose that E is a real Banach space, P is a cone in E and E is endowed with the partial ordering induced by P.

2.3 Definition: (L.G. Haung and X. Zhang [2])

Let X be a non-empty set. Suppose that the mapping d: X x X → P satisfies:

a. 0 < d (x, y) for all x, y ∈ X and d(x,y) = 0 if and only if $x = y$.

b. d(x, y) = d(y,x) for all $x, y \in$ X

c. d$(x, y) \leq$ d(x, z) + d(y, z) for all $x, y, z \in$ X.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

2.4 Definition: (L.G. Haung and X. Zhang [2])

Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X and $x \in$ X.

i. $\{x_n\}$ converges to x if for every c ∈ E with 0 << c, there is an n_o such that

for all $n > n_o$, $d(x_n, x) << c$. We denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$ as $n \to \infty$

ii. if for any c ∈ P with 0 << c, there is an n_o such that for all $n, m > n_o$,

$d(x_n, x_m) << c$, then $\{x_n\}$ is called a Cauchy sequence in X.

iii. (X, d) is called a complete cone metric space, if every Cauchy sequence in

X is convergent in X.

2.5 Definition: (H. Lakzian and F. Arabyani [6])

Let X be a cone metric space with metric d. Then a mapping $\omega : X\times X\rightarrow E$ is called ω - distance on X if the following conditions are satisfied

i) $0 \leq \omega(x, y)$ for all $x, y \in X$;

ii) $\boldsymbol{\omega}(x, z) \leq \omega(x, y) + \omega(y, z)$ for all $x, y, z \in X$;

iii) if $x_n \rightarrow x$ then $\omega(y,x_n) \rightarrow \omega(y,x)$ and $\omega(x_n, y) \rightarrow \omega(x,y)$

iv) for any $0 \ll \alpha$, there exists $0 \ll \beta$ such that $\omega(z, x) \ll \beta$ and $\omega(z, y) \ll \beta$ imply

$d(x, y) \ll \alpha$ for all $\alpha, \beta \in E$ … (2.5.1)

2.6 Definition: (H. Lakzian and F. Arabyani [6])

Let X be a cone metric space with metric d, let ω be a ω – distance on X, $x \in X$ and $\{x_n\}$ a sequence in X. Then $\{x_n\}$ is called a ω- Cauchy sequence whenever for every $\alpha \in E$, $0 \ll \alpha$, there is a positive integer N such that, for all $m,n \geq N$, $\omega(x_m,x_n) \ll \alpha$.

A sequence $\{x_n\}$ in X is called ω – convergent to a point $x \in X$ whenever for every $\alpha \in E$, $0 \ll \alpha$, there is a positive integer N such that for all $n \geq N$, $\omega(x,x_n) \ll \alpha$.

(X,d) is a complete cone metric space with ω- distance if every Cauchy Sequence is ω – Convergent.

2.7 Definition: (G. Jungck.and B.E. Rhoades [4])

Let S and T be self mappings of a set X. If $u = Sx = Tx$ for some $x \in X$, then x is called a coincidence point of S and T and u is called a point of coincidence of S and T.

2.8 Definition: (G. Jungck.and B.E. Rhoades [4])

Two self mappings S and T of a set X are said to be weakly compatible if they commute at their coincidence point. i.e; if $Su = Tu$ for some $u \in X$, then $STu = TSu$.

2.9 Proposition: (M.Abbas and G. Jungck [1])

Let S and T be weakly compatible self mappings of a set X. If S and T have a unique point of coincidence , i.e; $u = Sx = Tx$, then u is the unique common fixed point of S and T.

2.10 Property: Let (X,d) be a cone metric space. If $\{x_n\}$, $\{y_n\}$ are sequences in X and $x_n \to x$ and $y_n \to y$ then $d(x_n,y_n) \to d(x,y)$.

2.11 Assumption :

$x_n \to x$ (ie; $d(x_n,x) \to 0$)and $x_n \le y$ implies $x \le y$

3 Main Results

3.1 Lemma :

Let $\{y_n\}$ be a sequence in X such that

$$\omega(y_{n+1},y_n) \le \lambda\, \omega(y_n, y_{n-1}) \qquad \ldots(3.1.1)$$

where $0 < \lambda < 1$ then $\{y_n\}$is a Cauchy Sequence in X. Further, if $y_n \to y$ as $n \to \infty$, then $\omega(y,y) = 0$

Proof : We have by (3.1.1),

$$\omega(y_{n+1},y_n) \le \lambda^n w(y_1, y_0) , \qquad n = 1,2,3,\ldots$$

$\therefore$ for $n > m$,

$$\begin{aligned}\omega(y_n ,y_m) &\le \omega(y_n, y_{n-1}) + \omega(y_{n-1}, y_{n-2}) + \omega(y_{n-2}, y_{n-3}) + \ldots + \omega(y_{m+1}, y_m)\\ &\le \lambda^{n-1} \omega(y_1, y_0) + \lambda^{n-2} \omega(y_1, y_0) + \ldots + \lambda^{m} \omega(y_1, y_0)\\ &= [\lambda^{n-1} + \lambda^{n-2} + \lambda^{n-3} + \cdots + \lambda^{m}]\ \omega(y_1, y_0)\\ &= \left(\frac{\lambda^m}{1-\lambda} \right) \omega(y_1, y_0) \qquad \ldots(3.1.2)\\ &\to 0 \text{ as } m \to \infty\end{aligned}$$

Now let $0 \ll \eta$, choose δ according to (2.5.1).

By (3.1.2), $\omega(y_n, y_m) << \delta$ for m large

Hence $\omega(y_{n+1}, y_m) << \delta$ and $\omega(y_{n+1}, y_n) << \delta$

Hence $d(y_m, y_n) << \eta$ by (2.5.1)

Therefore $\{y_n\}$ is a Cauchy Sequence in (X,d)

Since $\omega(y_{n+1}, y_n) \leq \lambda^n \omega(y_1, y_0)$, $n = 1,2,\ldots$

we get $\omega(y_{m+k}, y_m) \leq \lambda^m \omega(y_1, y_0)$ …(3.1.3)

for large m and all k

Hence it converges to y , say.

In (3.1.3) , letting $k \to \infty$ we get

$$\omega(y , y_m) \leq \lambda^m \omega(y_1, y_0)$$

Now letting $m \to \infty$, $\omega(y , y_m) \to 0$

Hence $\omega(y , y) = 0$

3.2 Lemma:

If $w(x , y) = 0$ and $w(y , x) = 0$ then

(i) $w(x , x) = w(y , y) = 0$ and

(ii) $d(x,y) = 0$ so that $x = y$

Proof : Since $\omega(x , x) \leq \omega(x , y) + \omega(y , x) = 0$

we get $\omega(x , x) = 0$. Similarly $\omega(y , y) = 0$

Also we have $\omega(x , y) = 0$ so that $d(x,y) = 0$

Hence $x = y$

3.3 Theorem:

Let (X,d) be a complete cone metric space with ω – distance ω. Let P be a normal cone with normal constant K on X. Suppose that the mappings $S,T : X \to X$ satisfy the following conditions :

(i) The range of T contains the range of S and T(X) is a totally ordered closed subspace of X.

(ii) $\omega(Sx,Sy) \leq r\,[\,\omega(Sx,Ty)+\omega(Sy,Tx)+\omega(Sx,Tx)+\omega(Sy,Ty)+\max\{\omega(Tx,Ty),\omega(Ty,Tx)\}]$ …(3.3.1)

where $r \in [0,{}^{1}/_{7})$ is a constant. Then S and T have a unique coincidence point in X.

Moreover, if S and T are weakly compatible, then S and T have a unique common fixed point.

Proof : Let $x_0 \in X$. Since S(X) is contained in T(X), we choose a point x_1 in X such that $S(x_0) = T(x_1)$. Continuing this process we choose x_n and x_{n+1} in X such that $S(x_n) = T(x_{n+1})$.

Then $\omega(Tx_{n+1}, Tx_n) = \omega(Sx_n, Sx_{n-1})$

$\leq r\,[\,\omega(Sx_n,Tx_{n-1})+\omega(Sx_{n-1},Tx_n)+\omega(Sx_n,Tx_n)+\omega(Sx_{n-1},Tx_{n-1})+\max\{\omega(Tx_n,Tx_{n-1}),\omega(Tx_{n-1},Tx_n)\}]$

$= r\,[\,\omega(Tx_{n+1},Tx_{n-1})+\omega(Tx_n,Tx_n)+\omega(Tx_{n+1},Tx_n)+\omega(Tx_n,Tx_{n-1})+\max\{\omega(Tx_n,Tx_{n-1}),\omega(Tx_{n-1},Tx_n)\}]$

$\therefore\; w(Tx_{n+1}, Tx_n)$

$\leq r\,[\,\omega(Tx_{n+1},Tx_{n-1})+\omega(Tx_n,Tx_n)+\omega(Tx_{n+1},Tx_n)+\omega(Tx_n,Tx_{n-1})+\max\{\omega(Tx_n,Tx_{n-1}),\omega(Tx_{n-1},Tx_n)\}]$

Similarly $\omega(Tx_n, Tx_{n+1})$

$\leq r\,[\,\omega(Tx_{n+1},Tx_{n-1})+\omega(Tx_n,Tx_n)+\omega(Tx_{n+1},Tx_n)+\omega(Tx_n,Tx_{n-1})+\max\{\omega(Tx_n,Tx_{n-1}),\omega(Tx_{n-1},Tx_n)\}]$

$\therefore \max\{\omega(Tx_{n+1}, Tx_n), \omega(Tx_{n+1}, Tx_n)\}$

$\leq r\,[\omega(Tx_{n+1},Tx_{n-1})+\omega(Tx_n,Tx_n)+\omega(Tx_{n+1},Tx_n)+\omega(Tx_n,Tx_{n-1})+\max\{\omega(Tx_n,Tx_{n-1}),\omega(Tx_{n-1},Tx_n)\}]$

$\leq r\,[\omega(Tx_{n+1},Tx_n) + \omega(Tx_n,Tx_{n-1}) + \omega(Tx_n,Tx_{n-1}) + \omega(Tx_{n-1},Tx_n) + \omega(Tx_{n+1},Tx_n)+$

$\omega(Tx_n,Tx_{n-1})+\max\{\omega(Tx_n,Tx_{n-1}),\omega(Tx_{n-1},Tx_n)\}]$

Hence $\alpha_{n+1} \leq r\,[\,\alpha_{n+1} + \alpha_n + \alpha_n + \alpha_n + \alpha_{n+1} + \alpha_n + \alpha_n\,]$, where $\alpha_n = \max\{\omega(Tx_n,Tx_{n-1}),\omega(Tx_{n-1},Tx_n)\}$

$= r\,[\,2\,\alpha_{n+1} + 5\,\alpha_n\,]$

$\therefore \quad (1\text{-}2r)\, \alpha_{n+1} \leq 5\, r\, \alpha_n$

$\therefore \quad \alpha_{n+1} \leq \left(\frac{5r}{1-2r}\right) \alpha_n$

$\therefore \quad \alpha_n \to 0$ (since $\frac{5r}{1-2r} < 1$)

Hence { Tx_n } is a Cauchy Sequence in (X,d).

Since T(X) is a totally closed subspace of X, there exists q in T(X) such that $Tx_n \to q$ as $n \to \infty$.

Cosequently we can find h in X such that $T(h) = q$. Thus

$\omega(Tx_n, Sh) = \omega(Sx_{n-1}, Sh)$

$\leq r\,[\,\omega(Sx_{n-1},Th)+\omega(Sh,Tx_{n-1})+\omega(Sx_{n-1},Tx_{n-1})+\omega(Sh,Th)+\max\{\omega(Tx_{n-1},Th),\omega(Th,Tx_{n-1})\}]$

$= r\,[\,\omega(Tx_n,Th)+\omega w(Sh,Tx_{n-1})+\omega(Tx_n,Tx_{n-1})+\omega(Sh,Th)+\max\{\omega(Tx_{n-1},Th),\omega(Th,Tx_{n-1})\}]$

On letting $n \to \infty$

$\omega(Th, Sh) \leq r\,[\,\omega(Th,Th)+\omega(Sh,Th)+\omega(Th,Th)+\omega(Sh,Th)+\max\{\omega(Th,Th),\omega(Th,Th)\}]$

$= 2r\, \omega(Sh,Th)$

$\therefore \quad \omega(Th, Sh) \leq 2r\, \omega(Sh,Th)$.

Similarly $\omega(Sh, Th) \leq 2r[\,\omega(Sh,Th)\,]$

$\therefore \quad \omega(Th, Sh) = 0$ and $\omega(Sh,Th) = 0$

$\therefore \quad Sh = Th$ (by Lemma 3.2)

Hence h is a coincidence point of S and T

Uniqueness : Suppose that there exists a point u in X such that $Su = Tu$

So we have $\omega(Tu,Th) = \omega(Su,Sh)$

$\leq r\,[\,\omega(Su,Th)+\omega(Sh,Tu)+\omega(Su,Tu)+\omega(Sh,Th)+\max\{\omega(Tu,Th),\omega(Th,Tu)\}]$

$= r\,[\,\omega(Tu,Th)+\omega(Th,Tu)+\omega(Tu,Tu)+\omega(Th,Th)+\max\{\omega(Tu,Th),\omega(Th,Tu)\}]$

$\therefore$ ω(Tu,Th) ≤ r [ω(Tu,Th) + ω(Th,Tu) + max{ω(Tu,Th),ω(Th,Tu)}]

Similarly ω(Th,Tu) ≤ r [ω(Th,Tu) + ω(Tu,Th) + max{ω(Th,Tu),ω(Tu,Th)}]

$\therefore$ max {ω(Tu,Th),ω(Th,Tu)} ≤ r [ω(Th,Tu) + ω(Tu,Th) + max{ω(Th,Tu),ω(Tu,Th)}]

Suppose ω(Tu,Th) ≤ r [ω(Th,Tu) + ω(Tu,Th) + ω(Tu,Th)}]

so that (1-2r) ω(Tu,Th) ≤ r ω(Th,Tu)

$\therefore \omega(\mathrm{T}u,\mathrm{T}h) \leq \frac{r}{1-2r}\, \omega(\mathrm{T}h,\mathrm{Tu})$

Similarly $\omega(\mathrm{T}h,\mathrm{T}u) \leq \frac{r}{1-2r}\, \omega(\mathrm{T}u,\mathrm{Th})$

$\leq \frac{r^2}{(1-2r)^2}\, \omega(\mathrm{T}h,\mathrm{Tu})$

$\therefore \omega(\mathrm{T}h,\mathrm{T}u) > \omega(\mathrm{T}h,\mathrm{T}u)$ (since $\frac{r}{1-2r} > 1$) , a contradiction

$\therefore \omega(\mathrm{T}h,\mathrm{T}u) = 0$

Similarly $\omega(\mathrm{T}u,\mathrm{T}h) >> \omega(\mathrm{T}u,\mathrm{T}h)$, a contradiction

$\therefore \omega(\mathrm{T}u,\mathrm{T}h) = 0$

$\therefore$ Tu = Th

Hence h is a unique coincidence point of S and T.

Now suppose that S and T are weakly compatible.

Then, by Proposition 2.9 **,** h is the unique common fixed point of S and T.

Assuming that T(X) is totally ordered , the following result of Sami Ullah Khan and Arjamand Bano [8] follows as a Corollary.

3.4 Corollary: (Theorem 3.2 **,** Sami Ullah Khan and Arjamand Bano [8])

Let (X,d) be a complete cone metric space with ω – distance ω. Let P be a normal cone with normal constant K on X. Suppose that the mappings S,T : X → X satisfy the following conditions :

(i) The range of T contains the range of S and T(X) is a totally ordered closed subspace of X.

(ii) $\omega(Sx,Sy) \leq r\,[\,\omega(Sx,Ty) + \omega(Sy,Tx) + \omega(Sx,Tx) + \omega(Sy,Ty) + \omega(Tx,Ty)\,]$ …(3.4.1)

where $r \in [0, 1/7]$ is a constant. Then S and T have a unique coincidence point in X.

Moreover, if S and T are weakly compatible, then S and T have a unique common fixed point.

3.5 Remark:

In theorem 3.3, if $T = I_X$, the identity map on X , then as a consequence of theorem 3.3, we obtain the following result.

3.6 Corollary:

Let (X,d) be a complete cone metric space with ω – distance ω. Let P be a normal cone with normal constant K on X. Suppose that the mappings $S,T : X \to X$ satisfy the following conditions :

(i) The range of T contains the range of S and T(X) is a totally ordered closed subspace of X.

(ii) $\omega(Sx,Sy) \leq r\,[\,\omega(Sx,y)+\omega(Sy, x)+\omega(Sx, x)+\omega(Sy, y)+\max\{\omega(x,y),\omega(y,x)\}\,]$ …(3.6.1)

where $r \in [0, 1/7)$ is a constant. Then S and T have a unique coincidence point in X.

Moreover, if S and T are weakly compatible, then S and T have a unique common fixed point.

Acknowledgement :

The third author gratefully acknowledges

(i) the financial support from UGC under UGC Minor Research Project, NO.F: MRP – 4536/14 (SERO / UGC) in carrying on this research

(ii) the Principal Dr. Sr. N.D.Veronica and the management of St.Joseph's College for Women(A),Visakhapatnam for providing the necessary permission

(iii) the authorities of GITAM for providing necessary facilities to carry on this research.

References

[1] **M.Abbas and G. Jungck**, common fixed point results for non-commuting mappings without continuity in cone metric space, J. Math. Anal. Appl., 341(2008), 416-420.

[2] L.G Haung and X. Zhang, cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332(2) (2007), 1468-1476.

[3] D. IIić, V. Rakoćević, Common fixed point for maps on cone metric space. J. Math. Appl. 341 (2008) 876-882.

[4] G. Jungck.and B.E. Rhoades, fixed point for set valued functions without continuity, Indian J. Pure Appl. Math. 29(3)(1998), 227-238.

[5] O. Kada, T. Suzuki, W. Takahashi, non convex minimization theorems and fixed point theorems in complete metric spaces, Math. Japonica, 44(1996), 381-391.

[6] H. Lakzian and F. Arabyani, Some fixed point Theorems in cone metric spaces with w-Distance, Int. J. Math. Anal, Vol. 3 (22) (2009) 1081-1086.

[7] Sh. Rezapour, R. Hamlbarani, Some notes on the paper “Cone metric spaces and fixed point theorems of contractive mappings”, J. Math. Anal. Appl. 345 (2008) 719-724.

[8] Sami Ullah Khan and Arjamand Bano, Common Fixed Point Theorems in Cone Metric Space using W-Distance, Int. Journal of Math. Analysis, Vol. 7, 2013, no. 14, 657 - 663

CHAPTER 2

Fixed Point Theorems for a Self Map in Ordered Cone Metric Space Under Lattice Ordered c-Distance

1 Introduction

Huang and Zhang [13] introduced the concept of the cone metric space, replacing the set of real numbers by an ordered Banach space, and they showed some fixed point theorems of contractive type mappings on cone metric spaces. The Banach contraction principle is the most celebrated fixed point theorem [6]. Afterward, some various definitions of contractive mappings were introduced by other researchers and several fixed and common fixed point theorems were considered in [7, 10, 17, 19, 25]. Then, several fixed and common fixed point results in cone metric spaces were introduced in [2, 3, 9, 15, 24] and the references contained therein. Also, the existence of fixed and common fixed points in partially ordered cone metric spaces was studied in [4, 5, 27]. In 1996, Kada et al. [18] defined the concept of *w*-distance in complete metric space. Later, many authors proved some fixed point theorems in complete metric spaces (see [1, 20, 21, 23]). Also, note that Saadati et al. [26] introduced a probabilistic version of the *w*-distance of Kada et al. in a Menger probabilistic metric space. Recently, Cho et al. [8], and Wang and Guo [29] defined a concept of the c-distance in a cone metric space, which is a cone version of the *w*-distance of Kada et al. and proved some fixed point theorems in ordered cone metric spaces. Sintunavarat et al. [28] generalized the Banach contraction theorem on c-distance of Cho et al. [8]. Also, Dordevi´c et al. in [12] proved some fixed point and common fixed point theorems under c-distance for contractive mappings in tvs-cone metric spaces.

H. Rahimi, G. Soleimani Rad [22] extended the Banach contraction principle [6] and Chatterjea contraction theorem [7] on c-distance of Cho et al. [8], and proved some fixed point and common fixed point theorems in ordered cone metric spaces. In this paper we extend the results of [22]. Also we introduce the notion of lattice ordered c-distance and prove some fixed point theorems under a lattice ordered c-distance in ordered cone metric spaces, for a single function.

2 Preliminaries

First let us start with some basic definitions

Definition 2.1 ([13])

Let E be a real Banach space and P a subset of E. P is called a cone if

(i) P is closed, non-empty and $P \neq \{0\}$

(ii) $ax + by \in P \ \forall \ x, y \in P$ and non-negative real numbers a and b.

(iii) $P \cap (-P) = \{0\}$.

Definition 2.2 ([13])

We define a partial ordering $\leq$ on E with respect to P and $P \subset E$ by $x \leq y$ if and only if $y - x \in P$. We shall write $x << y$ if $y - x \in \text{int}\, P$, int P denotes the interior of P. We denote by $\|.\|$ the norm on E. The cone P is called normal if there is a number K > 0 such that for all $x, y \in E$, $0 \leq x \leq y$ implies

$$\|x\| \leq K\|y\|$$

The least positive number K satisfying (1.14.1) is called the normal constant of P.

Definition 2.3 ([13])

A cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}_{n\geq 1}$ is a sequence such that $x_1 \leq x_2 \leq \ldots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$

Definition 2.4 ([13])

Let X be a nonempty set and E be a real Banach space equipped with

the partial ordering $\leq$ with respect to the cone $P \subset E$. Suppose that the mapping

$d : X \times X \rightarrow E$ satisfies:

(d1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then, d is called a cone metric on X and (X,d) is called a cone metric space.

Definition 2.5 ([13])

Let (X,d) be a cone metric space, $\{x_n\}$ a sequence in X and $x \in$ X.

(i) $\{x_n\}$ converges to x if for every c $\in$ E with $0 \ll c$ there exists $n_0 \in N$ such that $d(x_n, x) \ll c$ for all $n > n_0$, and we write $\lim_{n\to\infty} d(x_n, x) = 0$.

(ii) $\{x_n\}$ is called a Cauchy sequence if for every c $\in$ E with $0 \ll c$ there exists $n_0 \in N$ such that $d(x_n, x_m) \ll c$ for all $m,n > n_0$, and we write $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.

(iii) If every Cauchy sequence in X is convergent, then X is called a complete cone metric space.

Lemma 2.6 ([13, 24])

Let (X,d) be a cone metric space and P be a normal cone with normal constant k. Also, let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y \in$ X. Then the following hold:

(c1) $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$.

(c2) If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then $x = y$.

(c3) If $\{x_n\}$ converges to x, then $\{x_n\}$ is a Cauchy sequence.

(c4) If $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$.

(c5) $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n,m \to \infty$.

Lemma 2.7 ([4, 14])

Let E be a real Banach space with a cone P in E. Then, for all u,v,w,c $\in$ E, the following hold:

(p1) If $u \le v$ and $v \ll w$, then $u \ll w$.

(p2) If $0 \le u \ll c$ for each $c \in \text{int } P$, then $u = 0$.

(p3) If $u \le \lambda u$ where $u \in P$ and $0 < \lambda < 1$, then $u = 0$.

(p4) Let $x_n \to 0$ in E, $0 \le x_n$ and $0 \ll c$. Then there exists positive integer n_0 such that $x_n \ll c$

for each $n > n_0$.

(p5) If $0 \le u \le v$ and k is a nonnegative real number, then $0 \le ku \le kv$.

(p6) If $0 \le u_n \le v_n$ for all $n \in N$ and $u_n \to u$, $u_n \to u$ as $n \to \infty$, then $0 \le u \le v$.

Definition 2.8 ([8, 29])

Let (X,d) be a cone metric space. A function $q : X \times X \to E$ is called a c-distance on X if the following are satisfied:

(q1) $0 \le q(x, y)$ for all $x, y \in X$;

(q2) $q(x, z) \le q(x, y) + q(y, z)$ for all $x, y, z \in X$;

(q3) for all $n \ge 1$ and $x \in X$, if $q(x, y_n) \le u$ for some u , then $q(x, y) \le u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$;

(q4) for all $c \in E$ with $0 \ll c$, there exists $e \in E$ with $0 \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

Remark 2.9 ([8, 29])

Each *w*-distance q in a metric space (X,d) is a c-distance (with $E = R^+$ and $P = [0,\infty)$). But the converse does not hold. Therefore, the c-distance is a generalization of *w*-distance.

Examples 2.10 ([8, 28, 29])

(1) Let (X,d) be a cone metric space and P be a normal cone. Put $q(x, y) = d(x, y)$ for all $x, y \in X$. Then q is a c-distance.

(2) Let $E = R$, $X = [0,\infty)$ and $P = \{x \in E : x \ge 0\}$. Define a mapping $d : X \times X \to E$ by $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a cone metric space. Define a mapping $q : X \times X \to E$ by $q(x, y) = y$ for all $x, y \in X$. Then q is a c distance.

(3) Let $E = C^1_R[0,1]$ with the norm $\|x\|= \|x\|_\infty+\|x'\|_\infty$ and consider the cone $P = \{x \in E : x(t) \ge 0 \text{ on } [0,1]\}$. Also, let $X = [0,\infty)$ and define a mapping $d : X\times X \to E$ by

$d(x, y) = |x-y|\psi$ for all $x, y \in X$, where $\psi : [0,1] \to R$ such that $\psi(t) = 2t$.

Then (X,d) is a cone metric space. Define a mapping $q : X \times X \to E$ by $q(x, y) = (x + y)\psi$ for all $x, y \in X$. Then q is c-distance.

(4) Let (X,d) be a cone metric space and P be a normal cone. Put $q(x, y) = d(w, y)$ for all $x, y \in X$, where $w \in X$ is a fixed point. Then q is a c-distance.

Remark 2.11 ([8, 28, 29])

From Examples 2.10 [1,2,4], we have three important results

(i) Each cone metric d on X with a normal cone is a c-distance q on X.

(ii) For c-distance q, $q(x, y) = 0$ is not necessarily equivalent to $x = y$ for all $x,y \in X$.

(iii) For c-distance q, $q(x,y) = q(y,x)$ does not necessarily hold for all $x,y \in X$.

Lemma 2.12 ([8, 28, 29])

Let (X,d) be a cone metric space and let q be a c-distance on X. Also, let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y, z \in X$. Suppose that $\{u_n\}$ and $\{v_n\}$ are two sequences in P converging to 0. Then the following hold:

(qp1) If $q(x_n, y) \preceq u_n$ and $q(x_n, z) \preceq v_n$ for $n \in N$, then $y = z$. Specifically, if $q(x, y) = 0$ and $q(x, z) = 0$, then $y = z$.

(qp2) If $q(x_n, y_n) \preceq u_n$ and $q(x_n, z) \preceq v_n$ for $n \in N$, then $\{y_n\}$ converges to z.

(qp3) If $q(x_n, x_m) \preceq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.

(qp4) If $q(y, x_n) \preceq u_n$ for $n \in N$, then $\{x_n\}$ is a Cauchy sequence in X.

The following special case of (qp3) plays a crucial role in determining Cauchy sequences, in Section 3

Lemma 2.13 :

In addition to the hypothesis of Lemma 2.12, assume the following :

Let z ∈ P , $0 \le \mu < 1$ and $q(x_n, x_m) \le \mu^n z_0 \ \forall \ m \ge n$.

Then $\{x_n\}$ is a Cauchy sequence. (∵ $u_n = \mu^n z_0 \longrightarrow 0$ as $n \longrightarrow \infty$)

Definition 2.14 ([4, 8])

Let (X, ≤) be a partially ordered set. Two mappings f, g : X → X are said to be weakly increasing if $f x \le g f x$ and $g x \le f g x$ hold for all $x \in$ X.

Definition 2.15 ([9])

A lattice is a partially ordered set S in which any two elements a,b ∈ S have the supremum (a ∪ b) and the infimum (a ∩ b). Sometimes we write max { a,b } for (a ∪ b) and min { a,b } for (a ∩ b).

H. Rahimi, G. Soleimani Rad [22] proved following theorem.

Theorem 2.16 : ([22] , Theorem 3.1)

Let (X, ≤) be a partially ordered set and (X,d) be a complete cone metric space. Also, let q be a c-distance on X and f : X → X be a continuous and non decreasing mapping with respect to ≤. Suppose that there exist mappings α,β,γ : X → [0,1) such that the following four conditions hold:

(i) $\alpha(fx) \le \alpha(x)$, $\beta(fx) \le \beta(x)$ and $\gamma(fx) \le \gamma(x)$ for all $x \in X$ … (2.16.1)

(ii) $(\alpha + 2\beta + 2\gamma)(x) < 1$ for all $x \in X$ … (2.16.2)

(iii) for all $x,y \in X$ with $x \le y$, $q(fx,fy) \preceq \alpha(x)q(x,y) + \beta(x)q(x,fy) + \gamma(x)q(y,fx)$ … (2.16.3)

(iv) for all $x,y \in X$ with $x \le y$, $q(fy,fx) \preceq \alpha(x)q(y,x) + \beta(x)q(fy,x) + \gamma(x)q(fx,y)$ … (2.16.4)

If there exists $x_0 \in X$ such that $x_0 \le fx_0$, then f has a fixed point. Moreover, if $fz = z$, then $q(z,z) = 0$.

Note : In the above theorem α, β, γ are functions of x. In section 3, we take α, β, γ to be constants so that above (i) becomes obvious.

3 Main results

In this section , we extend the result of H. Rahimi, G. Soleimani [22]. Also we introduce the notion of lattice ordered c-distance and prove some fixed point theorems under a lattice ordered c-distance in ordered cone metric spaces, for a single function.

Let us introduce the notion of a lattice ordered c-distance in cone metric spaces.

Definition : Suppose (X,d) is a cone metric space and q : X × X → E is a c-distance. Clearly, the image q (X × X) of X × X under q is a subset of P. If the image q (X × X) is a lattice in P, we say that q is a lattice ordered c-distance on X.

Now we state and prove two of our main results with the underlying space X having a lattice ordered c-distance. This becomes necessary since we consider maximum of three terms in the control function which we cannot do if the c-distance is not lattice ordered.

Theorem 3.1

Let $(X, \leq)$ be a partially ordered set and (X, d) be a complete cone metric space. Also, let q be a lattice ordered c-distance on X and f : X → X be continuous non decreasing mapping with respect to ≤ . Suppose $\lambda \in [\,0, \frac{1}{2}\,)$ and q satisfies the following conditions :

(i) $x \leq y \leq z$ implies $q(x, y) \leq q(x, z)$ and $q(y, z) \leq q(x, z)$ $\forall x, y, z \in X$ …(3.1.1)

(ii) $q(fx, fy) \leq \lambda \max\{q(x, y), q(x, fy), q(y, fx)\}$ $if\ x \leq y$ ….(3.1.2)

Suppose there exists $x_0 \in X$ such that $x_0 \leq f x_0$. Then f has a fixed point in X. Moreover, if $fz = z$, then $q(z,z) = 0$.

Proof : Write $x_n = f x_{n-1}$, n = 1,2,3,…

If $x_n = x_{n+1}$ for some n, then $x_n = x_{n+1} = f x_n$ so that x_n is a fixed point of f . Now,

suppose that $fx_0 \neq x_0$. Since f is non decreasing with respect to ≤ and $x_0 \leq fx_0$,

we get $f x_0 \leq f^2 x_0 \Longrightarrow x_1 \leq x_2$

In a similar way we can show that $x_n \leq x_{n+1}$ ∀ n = 0,1,2,…

Here $x_n = f x_{n-1} = f^n x_0$

Now, let $x = x_n$ and $y = x_{n-1}$ in (3.1.2), we have

$\therefore\ q\ (x_n, x_{n+1})$

$= q\ (fx_{n-1}, fx_n)$

$\leq \lambda\ max\ \{\ q\ (x_{n-1}, x_n), q\ (x_{n-1}, fx_n), q\ (x_n, fx_{n-1})\}\ \ (\because x_{n-1} \leq x_n\ )$

$= \lambda\ max\ \{\ q\ (x_{n-1}, x_n), q\ (x_{n-1}, x_{n+1}), q\ (x_n, x_n)\}$

$= \lambda\ q\ (x_{n-1}, x_{n+1})$ (from (3.1.1))

$\leq \lambda\ \{\ q\ (x_{n-1}, x_n) + q\ (x_n, x_{n+1})\ \}$

$\Longrightarrow q\ (x_n, x_{n+1}) \leq (\frac{\lambda}{1-\lambda})\ q\ (x_{n-1}, x_n)$

$\therefore$ By induction ,

$$q\ (x_n, x_{n+1}) \leq (\frac{\lambda}{1-\lambda})^{n-1}\ q\ (x_0\ , x_1) \quad \forall\ n \in N$$

Now $\frac{\lambda}{1-\lambda} < 1\ \ since\ \ 0 \leq \lambda < ½$

$\therefore$ By taking $\mu = \frac{\lambda}{1-\lambda}$ and $z_0 = q\ (x_0\ , x_1) \in P$, from Lemma 2.13, we get that $\{\ x_n\ \}$ is a Cauchy sequence in X. Since X is complete, there exists a point z ∈ X such that $x_n \to z$ as $n \to \infty$. Continuity of f implies that $x_{n+1} = fx_n \to fz$ as $n \to \infty$ and since the limit of a sequence is unique, we get that $fz = z$. Thus, z is a fixed point of f.

since $fz = z$, $q\ (z, z)\ = q\ (fz, fz)$

$\leq \lambda\ max\ \{\ q\ (z, z), q\ (z, fz), q\ (z, fz)\}$

$= \lambda\ max\ \{\ q\ (z, z), q\ (z, z), q\ (z, z)\}$

$= \lambda\ q\ (z, z)$

$\Longrightarrow\ q\ (z, z) = 0$

Theorem 3.2 : Let $(X , \leq)$ be a partially ordered set and (X, d) be a complete cone metric space. Also, let q be a lattice ordered c-distance on X and $f : X \to X$ be continuous non decreasing mapping with respect to $\leq$. Suppose $\lambda \in [\ 0,\ ½\ )$ and q satisfies the following conditions :

(i) $x \leq y \leq z$ implies $q(y,x) \leq q(z,x)$ and $q(z,y) \leq q(z,x)\ \forall x,y,z \in X$ …(3.2.1)

(ii) $q(fy,fx) \leq \lambda \max\{q(y,x)\ ,q(fy,x)\ ,\ q(fx,y)\}\ \ if\ x \leq y$ ….(3.2.2)

Suppose there exists $x_0 \in X$ such that $x_0 \leq f x_0$. Then f has a fixed point in X.

Moreover, if $f z = z$, then q(z,z) = 0.

Proof : As in Theorem 3.1 , write $x_n = f x_{n-1}$, n = 1,2,3,…

Then $x_n \leq x_{n+1}$ for n = 0,1,2,… since f is non- decreasing

Here $x_n = f x_{n-1} = f^n x_0$

Now, let $x = x_{n-1}$ and $y = x_n$ in (3.1.2), we have

$\therefore\ \ q\ (x_{n+1}, x_n)$

$=\ q\ (fx_n, fx_{n-1})$

$\leq \lambda\ max\ \{\ q\ (x_n, x_{n-1}), q\ (fx_n, x_{n-1}), q\ (fx_{n-1}, x_n)\}$ ($\because$ by (3.2.1))

$=\ \lambda\ max\ \{\ q\ (x_n, x_{n-1}), q\ (x_{n+1}, x_{n-1}), q\ (x_n, x_n)\}$

$=\ \lambda\ \ q\ (x_{n+1}, x_{n-1})$ (from 3. 2.1)

$\leq\ \lambda\ \{\ q\ (x_{n+1}, x_n)\ +\ q\ (x_n, x_{n-1})\ \}$

$\Longrightarrow q\ (x_{n+1}, x_n)\ \ \leq\ (\frac{\lambda}{1-\lambda})\ q\ (x_n, x_{n-1})$

$\therefore$ By induction ,

$q\ (x_{n+1}, x_n)\ \ \leq\ (\frac{\lambda}{1-\lambda})^{n-1}\ q\ (x_1\ , x_0)$ for n = 1,2,3,…

Now $\frac{\lambda}{1-\lambda} < 1$ since $0 \leq \lambda < ½$

$\therefore$ By Lemma 2.13 , $\{\ x_n\ \}$ is a Cauchy sequence in X. Since X is complete, there exists a

point $z \in X$ such that $x_n \to z$ as $n \to\ \infty$. Continuity of f implies that $x_{n+1} = fx_n \to f z$ as

$n \to \infty$ so that $fz = z$. Thus, z is a fixed point of f.

since $fz = z$, $q(z,z) = q(fz, fz)$

$$\leq \lambda\, max\, \{ q(z,z), q(fz,z), q(fz,z) \}$$

$$= \lambda\, max\, \{ q(z,z), q(z,z), q(z,z) \}$$

$$= \lambda\, q(z,z)$$

$$\Longrightarrow q(z,z) = 0.$$

In the following theorem we obtain a condition under which a function may admit unique fixed point.

Theorem 3.3 : Let $(X, \leq)$ be a partially ordered set and (X, d) be a complete cone metric space. Also, let q be a lattice ordered c-distance on X and $f : X \to X$ be continuous non decreasing mapping with respect to $\leq$. Suppose $\lambda \in [0, ½)$ and q satisfies the following conditions :

(i) $x \leq y \leq z$ implies $q(x,y) \leq q(x,z)$; $q(y,x) \leq q(z,x)$ …(3.3.1)

$q(y,z) \leq q(x,z)$; $q(z,y) \leq q(z,x)$ $\forall\, x,y,z \in X$ … (3.3.2)

(ii) $q(fx,fy) \leq \lambda\, max\{q(x,y), q(x,fy), q(y,fx)\}$ $if\ x \leq y$ … (3.3.3)

(iii) $q(fy,fx) \leq \lambda\, max\{q(y,x), q(fy,x), q(fx,y)\}$ $if\ x \leq y$ ...(3.3.4)

Suppose there exists $x_0 \in X$ such that $x_0 \leq f x_0$. Then f has a fixed point in X and no two fixed points are comparable. Moreover, if $f z = z$, then $q(z,z) = 0$.

Proof : Write $x_n = f x_{n-1}$, n = 1,2,3,…

If $x_n = x_{n+1}$ for some n, then $x_n = x_{n+1} = f x_n$ so that x_n is a fixed point of f . Now, suppose that $f x_0 \neq x_0$. Since f is non decreasing with respect to $\leq$ and $x_0 \leq fx_0$,

we get $f x_0 \leq f^2 x_0 \Longrightarrow x_1 \leq x_2$

In a similar way we can show that $x_n \leq x_{n+1}$ $\forall$ n = 0,1,2,…

Here $x_n = f x_{n-1} = f^n x_0$

Now, let $x = x_n$ and $y = x_{n-1}$ in (3.3.3), we have

$\therefore\ q\ (x_n, x_{n+1})$

$=\ q\ (fx_{n-1}, fx_n)$

$\leq \lambda\ max\ \{\ q\ (x_{n-1}, x_n), q\ (x_{n-1}, fx_n), q\ (x_n, fx_{n-1})\}\ (\because x_{n-1}\ \leq\ x_n\ )$

$= \lambda\ max\ \{\ q\ (x_{n-1}, x_n), q\ (x_{n-1}, x_{n+1}), q\ (x_n, x_n)\}$

$=\ \lambda\ \ q\ (x_{n-1}, x_{n+1})$ (from 3.3.1)

$\leq\ \lambda\ \{\ q\ (x_{n-1}, x_n) +\ q\ (x_n, x_{n+1})\ \}$

$\Longrightarrow q\ (x_n, x_{n+1})\ \ \leq\ (\frac{\lambda}{1-\lambda})\ q\ (x_{n-1}, x_n)$

$\therefore$ By induction ,

$$q\ (x_n, x_{n+1})\ \ \leq\ (\frac{\lambda}{1-\lambda})^{n-1}\ q\ (x_0\ , x_1)$$

Similarly, using (3.3.4), we get

$q\ (x_{n+1}, x_n)$

$=\ q\ (fx_n, fx_{n-1})$

$\leq \lambda\ max\ \{\ q\ (x_n, x_{n-1}), q\ (fx_n, x_{n-1}), q\ (fx_{n-1}, x_n)\}$ ($\because$ by (3.2.1))

$=\ \lambda\ max\ \{\ q\ (x_n, x_{n-1}), q\ (x_{n+1}, x_{n-1}), q\ (x_n, x_n)\}$

$=\ \lambda\ \ q\ (x_{n+1}, x_{n-1})$ (from 3. 2.1)

$\leq\ \lambda\ \{\ q\ (x_{n+1}, x_n) + q\ (x_n, x_{n-1})\ \}$

$\Longrightarrow q\ (x_{n+1}, x_n)\ \ \leq\ (\frac{\lambda}{1-\lambda})\ q\ (x_n, x_{n-1})$

$\therefore$ By induction ,

$$q\ (x_{n+1}, x_n)\ \ \leq\ (\frac{\lambda}{1-\lambda})^{n-1}\ q\ (x_1\ , x_0)\ \ \text{for n} = 1,2,3,\ldots$$

Now $\frac{\lambda}{1-\lambda} < 1\ \ since\ \ 0 \leq \lambda < ½$

∴ By Lemma 2.11 , { x_n } is a Cauchy sequence in X. Since X is complete, there exists a point z ∈ X such that $x_n \to z$ as n → ∞ .

∴ By continuity of f we get

$x_{n+1} = fx_n \to f$z as n → ∞ .

Hence *f*z = *z*. Thus, *z* is a fixed point of *f*.

Now $q\ (z,z) = q\ (fz,fz)$

$\leq \lambda\, max\ \{ q\ (z,z), q\ (z,fz), q\ (z,fz)\}$

$= \lambda\, max\ \{ q\ (z,z), q\ (z,z), q\ (z,z)\}$

$= \lambda\, q\ (z,z)$

$\Longrightarrow q\ (z,z) = 0$

Uniqueness : Suppose z and z′ are two comparable fixed points of f so that *f*z = z and *f*z′ = z′

We may suppose without loss of generality that $z \leq z'$

Now $q\ (z,z')$

$= q\ (fz,fz')$

$\leq \lambda\, max\ \{ q\ (z,z'), q\ (z,fz'), q\ (z',fz)\}$ (since $z \leq z'$ from (3.3.3))

$= \lambda\, max\ \{ q\ (z,z'), q\ (z,z'), q\ (z',z)\}$

$= \lambda\, max\{ q\ (z,z'), q\ (z',z)\}$

$\therefore\ q\ (z,z') \leq \lambda\, max\{q\ (z,z'), q\ (z',z)\}$ …(3.3.5)

Also $q\ (z',z) = q\ (fz',fz)$

$\leq \lambda\, max\ \{ q\ (z',z), q\ (fz',z), q\ (fz,z')\}$ (since $z \leq z'$ from (3.3.4))

$= \lambda\, max\ \{ q\ (z',z), q\ (z',z), q\ (z,z')\}$

$$= \lambda \, max \{ q \, (z', z), q \, (z, z') \}$$

$$\therefore \;\; q \, (z', z) \;\; \leq \;\; \lambda \, max \{ q \, (z', z), q \, (z, z') \} \qquad \ldots (3.3.6)$$

Suppose $\max\{ q \, (z, z'), q \, (z', z)\} \;\; \neq \; 0$

Then from (3.3.5) and (3.3.6)

$$\max \{ q \, (z, z'), q \, (z', z)\} \; \leq \;\; \lambda \, max \; \{ q \, (z', z), q \, (z, z')\}$$

$$< \;\; max \, \{ q \, (z', z), q \, (z, z')\} \text{ , a contradiction}$$

Hence $\max\{ q \, (z', z), q \, (z, z')\} \; = 0$

Consequently, $q \, (z', z) \; = 0 = q \, (z, z')$

$$\therefore \qquad z = z'$$

Hence f cannot have two comparable fixed points.

The following theorem which is an analogue of theorem 3.1, for decreasing functions can be easily established.

Theorem 3.4 : Let $(X, \leq)$ be a partially ordered set and (X, d) be a complete cone metric space. Also, let q be a lattice ordered c-distance on X and $f : X \rightarrow X$ be continuous non decreasing mapping with respect to $\leq$. Suppose $\lambda \in [\; 0, ½ \;)$ and q satisfies the following conditions :

(i) $x \leq y \leq z$ implies $q(x, y) \; \leq q(x, z)$ and $q(y, z) \; \leq q(x, z) \; \forall x, y, z \in X$ …(3.4.1)

(ii) $q(fx, fy) \; \leq \; \lambda \max\{q(x, y) \, , q(x, fy) \, , \; q(fx, y)\} \;\; if \; x \; \leq y$ ….(3.4.2)

Suppose there exists $x_0 \in X$ such that $x_0 \geq f \, x_0$. Then f has a fixed point in X. Moreover and if $f z = z$, then $q(z,z) = 0$.

The following theorem which is an analogue of theorem 3.2, for decreasing functions can be easily established.

Theorem 3.5 : Let $(X, \leq)$ be a partially ordered set and (X, d) be a complete cone metric space. Also, let q be a lattice ordered c-distance on X and $f : X \rightarrow X$ be continuous non decreasing mapping with respect to $\leq$. Suppose $\lambda \in [\ 0,\ ½\ )$ and q satisfies the following conditions :

(i) $x \leq y \leq z$ implies $q(y,x) \leq q(z,x)$ and $q(z,y) \leq q(z,x)$ $\forall x,y,z \in X$ …(3.5.1)

(ii) $q(fy,fx) \leq \lambda \max\{q(y,x), q(fy,x), q(y,fx)\}$ $if\ x \leq y$ ….(3.5.2)

Suppose there exists $x_0 \in X$ such that $x_0 \geq f x_0$. Then f has a fixed point in X. Moreover, if $f z = z$, then $q(z,z) = 0$.

Now we prove an improved version of theorem 2.16

Theorem 3.6 : Let (X,≤) be a partially ordered set and (X,d) be a complete cone metric space. Also, let q be a c-distance on X and $f : X \to X$ be a continuous and non decreasing mapping with respect to ≤. Suppose $x_0 \in X$ such that $x_0 \leq fx_0$. Write $x_n = f^n x_0$, $n =$ 0,1,2, ... Suppose there exist α,β,γ with $\alpha + 2\beta + 2\gamma < 1$ (non negative constants) such that $q(x_{n+1}, x_{m+1}) \leq \alpha q(x_n, x_m) + \beta q(x_n, x_{m+1}) + \gamma q(x_m, x_{n+1})$ …(3.6.1)

$q(x_{m+1}, x_{n+1}) \leq \alpha q(x_m, x_n) + \beta q(x_{m+1}, x_n) + \gamma q(x_{n+1}, x_m)$ …(3.6.2)

for n = 0,1,2,… and m > n.

Then $\{x_n\}$ is a Cauchy sequence with limit z (say) and z is a fixed point of f.

Moreover, if $fz = z$, then $q(z,z) = 0$.

Proof : Write $x_n = f x_{n-1}$, n = 1,2,3,…

If $x_n = x_{n+1}$ for some n, then $x_n = x_{n+1} = f x_n$ so that x_n is a fixed point of f .Now,

suppose that $fx_0 \neq x_0$. Since f is non decreasing with respect to ≤ and $x_0 \leq fx_0$,

we get $f x_0 \leq f^2 x_0 \Longrightarrow x_1 \leq x_2$

In a similar way we can show that $x_n \leq x_{n+1}$ ∀ n = 0,1,2,…

Here $x_n = f x_{n-1} = f^n x_0$

Now, let $x_n = x_n$ and $x_m = x_{n+1}$ in (3.6.1), we have

$q(x_{n+1}, x_{n+2}) = q(fx_n, fx_{n+1})$

$\leq \alpha q(x_n, x_{n+1}) + \beta q(x_n, x_{n+2}) + \gamma q(x_{n+1}, x_{n+1})$

$\leq \alpha q(x_n, x_{n+1}) + \beta\{q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2})\} +$

$\gamma\{q(x_{n+1}, x_n) + q(x_n, x_{n+1})\}$

$= (\alpha + \beta + \gamma)\, q(x_n, x_{n+1}) + \beta\, q(x_{n+1}, x_{n+2}) + \gamma\, q(x_{n+1}, x_n)$

$\therefore q(x_{n+1}, x_{n+2}) \leq (\alpha + \beta + \gamma)\, q(x_n, x_{n+1}) + \beta\, q(x_{n+1}, x_{n+2}) + \gamma\, q(x_{n+1}, x_n)$ …(3.6.3)

Now $q(x_{n+2}, x_{n+1})$

$= q(fx_{n+1}, fx_n)$

$\leq \alpha q(x_{n+1}, x_n) + \beta q(x_{n+2}, x_n) + \gamma q(x_{n+1}, x_{n+1})$ (from (3.6.2))

$\leq \alpha q(x_{n+1}, x_n) + \beta\{q(x_{n+2}, x_{n+1}) + q(x_{n+1}, x_n)\} +$

$\gamma\{q(x_{n+1}, x_n) + q(x_n, x_{n+1})\}$

$= (\alpha + \beta + \gamma)\, q(x_{n+1}, x_n) + \beta\, q(x_{n+2}, x_{n+1}) + \gamma\, q(x_n, x_{n+1})$

$\therefore q(x_{n+2}, x_{n+1}) \leq (\alpha + \beta + \gamma)\, q(x_{n+1}, x_n) + \beta\, q(x_{n+2}, x_{n+1}) + \gamma\, q(x_n, x_{n+1})$ ...(3.6.4)

Combining (3.6.3) and (3.6.4) we get

$q(x_{n+1}, x_{n+2}) + q(x_{n+2}, x_{n+1}) \leq (\alpha + \beta + \gamma)\{q(x_n, x_{n+1}) + q(x_{n+1}, x_n)\}$

$+\beta\,\{q(x_{n+1}, x_{n+2}) + q(x_{n+2}, x_{n+1}) + \gamma\{\, q(x_{n+1}, x_n) + q(x_n, x_{n+1})\}$

$\Longrightarrow \lambda_n \leq (\alpha + \beta + \gamma)\, \lambda_{n-1} + \beta\, \lambda_n + \gamma\, \lambda_{n-1}$ where

$\lambda_n = q(x_{n+1}, x_{n+2}) + q(x_{n+2}, x_{n+1})$

$\Longrightarrow \quad \lambda_n \leq \left(\frac{\alpha+\beta+2\gamma}{1-\beta}\right) \lambda_{n-1}$

$\therefore \quad \lambda_n \leq \mu\, \lambda_{n-1}$, n = 1,2,3,… where $\mu = \frac{\alpha+2\beta+2\gamma}{1-\beta} < 1$

$\therefore$ By Lemma 2.11, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$. Continuity of f implies that $x_{n+1} = fx_n \to fz$ as $n \to \infty$ so that $fz = z$. Thus, z is a fixed point of f.

Since $fz = z$, $q\ (z, z)\ = q\ (fz, fz)$

$$\leq \alpha q(z,z) + \beta q(z,z) + \gamma q(z,z)$$

$$= (\alpha + \beta + \gamma)\ q(z,z)$$

$$< q(z,z)\ (\ \alpha + \beta + \gamma < \alpha + 2\beta + 2\gamma < 1\ )$$

$$\Longrightarrow q\ (z,z) = 0.$$

Now we show that theorem 2.16 is a simple consequence of theorem 3.6.

Corollary 3.7 : (H. Rahimi , G. Soleimani Rad [22], Theorem 3.1)

Let $(X,\leq)$ be a partially ordered set and (X,d) be a complete cone metric space. Also, let q be a c-distance on X and $f : X \to X$ be a continuous and non decreasing mapping with respect to $\leq$. Suppose that there exist mappings $\alpha,\beta,\gamma : X \to [0,1)$ such that the following four conditions hold:

(i) $\alpha(fx) \leq \alpha(x)$, $\beta(fx) \leq \beta(x)$ and $\gamma(fx) \leq \gamma(x)$ for all $x \in X$;

(ii) $(\alpha + 2\beta + 2\gamma)(x) < 1$ for all $x \in X$;

(iii) for all $x,y \in X$ with $x \leq y$, $q(fx,fy) \preceq \alpha(x)q(x,y) + \beta(x)q(x,fy) + \gamma(x)q(y,fx)$;

(iv) for all $x,y \in X$ with $x \leq y$, $q(fy,fx) \preceq \alpha(x)q(y,x) + \beta(x)q(fy,x) + \gamma(x)q(fx,y)$.

Suppose there exists $x_0 \in X$ such that $x_0 \leq fx_0$. Then f has a fixed point in X.

Moreover, if $fz = z$, then $q(z,z) = 0$.

Proof : Let $\alpha = \alpha(x_0)$, $\beta = \beta(x_0)$, $\gamma = \gamma(x_0)$

Then take $x = x_n$, $y = x_m$ in Theorem 3.6 we get

$q(x_{n+1}, x_{m+1})$

$= q(fx_n, fx_m)$

$\leq \alpha(x_n)q(x_n, x_m) + \beta(x_n)q(x_n, fx_m) + \gamma(x_n)q(x_m, fx_n)$

$= \alpha(x_n)q(x_n, x_m) + \beta(x_n)q(x_n, x_{m+1}) + \gamma(x_n)q(x_m, x_{n+1})$

$\leq \alpha(x_0)q(x_n, x_m) + \beta(x_0)q(x_n, x_{m+1}) + \gamma(x_0)q(x_m, x_{n+1})$

($\because \alpha(x_n) \leq \alpha(x_0)$ for n = 1,2,3,…)

$= \alpha\, q(x_n, x_m) + \beta\, q(x_n, x_{m+1}) + \gamma\, q(x_m, x_{n+1})$ …(3.7.1)

where $\alpha = \alpha(x_0)$, $\beta = \beta(x_0)$ and $\gamma = \gamma(x_0)$

Similarly we get $q(x_{m+1}, x_{n+1})$

$\leq \alpha\, q(x_m, x_n) + \beta\, q(x_{m+1}, x_n) + \gamma\, q(x_{n+1}, x_m)$ …(3.7.2)

Combining (3.7.1) and (3.7.2) follows that the hypothesis of Theorem 3.6 is satisfied. Hence from theorem 3.6 the result follows.

The following theorem establishes a condition for the uniqueness of the fixed point.

Theorem 3.8 : Under the hypothesis of Corollary 3.7 , no two fixed points are comparable.

Proof : Suppose z and z′ are two comparable fixed points of f so that $fz = z$ and $fz' = z'$

We may suppose without loss of generality that $z \leq z'$

Now $q(z, z') = q(fz, fz')$

$\leq \alpha(z)q(z, z') + \beta(z)q(z, fz') + \gamma(z)q(z', fz)$

$= \alpha\, q(z, z') + \beta\, q(z, z') + \gamma\, q(z', z)$ where $\alpha = \alpha(z), \beta = \beta(z), \gamma = \gamma(z)$

$= (\alpha + \beta)\, q(z, z') + \gamma\, q(z', z)$

$\therefore\ q(z, z') \leq \left(\frac{\gamma}{1-\alpha-\beta}\right) q(z', z)$ …(3.8.1)

Again $q(z', z) = q(fz', fz)$

$\leq \alpha(z)q(z', z) + \beta(z)q(fz', z) + \gamma(z)q(fz, z')$

$= \alpha\, q(z', z) + \beta\, q(z', z) + \gamma\, q(z, z')$

$= (\alpha + \beta)\, q(z', z) + \gamma\, q(z, z')$

$\therefore\ q(z', z) \leq \left(\frac{\gamma}{1-\alpha-\beta}\right) q(z, z')$ …(3.8.2)

From (3.8.1) and (3.8.2)

$q\ (z, z') \ \leq (\frac{\gamma}{1-\alpha-\beta})^2\ q(z, z')$

$q\ (z, z') = 0 \qquad (\because \ (\frac{\gamma}{1-\alpha-\beta}) < 1)$

similarly $q\ (z', z) = 0$

$\therefore \ q\ (z, z') = 0 = q\ (z', z)$

Hence $z = z'$

Hence f cannot have two comparable fixed points.

Now the following theorem which differs from Theorem 2.16 in the conditions (2.16.3) and (2.16.4) compared to (3.9.3) and (3.9.4) (in the last terms) can be easily established.

Theorem 3.9 : Let (X,≤) be a partially ordered set and (X,d) be a complete cone metric space. Also, let q be a c-distance on X and $f : X \rightarrow X$ be a continuous and non decreasing mapping with respect to ≤. Suppose that there exist mappings $\alpha,\beta,\gamma : X \rightarrow [0,1)$ such that the following four conditions hold:

(i) $\alpha(fx) \leq \alpha(x)$, $\beta(fx) \leq \beta(x)$ and $\gamma(fx) \leq \gamma(x)$ for all $x \in X$ …(3.9.1)

(ii) $(\alpha + 2\beta + 2\gamma)(x) < 1$ for all $x \in X$ …(3.9.2)

(iii) for all $x, y \in X$ with $x \leq y$, $q(fx,fy) \preceq \alpha(x)q(x,y) + \beta(x)q(x,fy) + \gamma(x)q(fx,y)$ …(3.9.3)

(iv) for all $x,y \in X$ with $x \leq y$, $q(fy,fx) \preceq \alpha(x)q(y,x) + \beta(x)q(fy,x) + \gamma(x)q(y,fx)$ …(3.9.4)

If there exists $x_0 \in X$ such that $x_0 \leq fx_0$, then f has a fixed point and no two fixed points are comparable. Moreover, if $fz = z$, then $q(z,z) = 0$.

Acknowledgement : The third author gratefully acknowledges

(i) the financial support from UGC under UGC Minor Research Project, NO.F: MRP – 4536/14 (SERO / UGC) in carrying on this research

(ii) the Principal Dr. Sr. Shyji and the management of St.Joseph's College for

Women(A),Visakhapatnam for providing the necessary permission

(iii) the authorities of GITAM for providing necessary facilities to carry on this research.

References

[1] **M. Abbas, D. Ili´c, M. Ali Khan**, Coupled coincidence point and coupled common fixed point theorems in partially ordered metric spaces with w-distance, Fixed Point Theory and Applications 11 (2010) Article ID 134897.

[2] **M. Abbas, G. Jungck**, Common fixed point results for noncommuting mappings without continuity in cone metric spaces, Journal of Mathematical Analysis and Applications 341 (2008) 416-420.

[3] **M. Abbas, B. E. Rhoades, T. Nazir**, Common fixed points for four maps in cone metric spaces, Applied Mathematics and Computation 216 (2010) 80-86.

[4] **I. Altun, B. Damnjanovi´c, D. Djori´c,** Fixed point and common fixed point theorems on ordered cone metric spaces, Applied Mathematics Letters 23 (2010) 310-316.

[5] **I. Altun, G. Durmaz**, Some fixed point theorems on ordered cone metric spaces, Rendiconti del Circolo Matematico di Palermo 58 (2009) 319-325.

[6] **S. Banach**, Sur les oprations dans les ensembles abstraits et leur application aux equations intgrales, Fundamenta Mathematicae 3 (1922) 133-181.

[7] **S. K. Chatterjea**, Fixed point theorems, Comptes rendus de l'Academie bulgare des Sciences 25 (1972) 727-730.

[8] **Y. J. Cho, R. Saadati, S. H. Wang**, Common fixed point theorems on generalized distance in ordered cone metric spaces, Computers and Mathematics with Applications 61 (2011) 1254-1260.

[9] **B. S. Choudhury, N. Metiya**, The point of coincidence and common fixed point for a pair of mappings in cone metric spaces, Computers and Mathematics with Applications 60 (2010) 1686-1695.

[10] **L. B. ´Ciri´c,** A generalization of Banach's contraction principle, Proceedings of the American Mathematical Society 45 (1974) 267-273.

[11] **K. Deimling**, Nonlinear Functional Analysis, Springer-Verlag, (1985).

[12] **M. Dordevi´c, D. Dori´c, Z. Kadelburg, S. Radenovi´c, D. Spasi´c,** Fixed point results under c-distance in tvs-cone metric spaces, Fixed Point Theory and Applications (2011) http://dx.doi.org/10.1186/ 1687-1812-2011-29/.

[13] **L. G. Huang, X. Zhang,** Cone metric spaces and fixed point theorems of contractive mappings, Journal of Mathematical Analysis and Applications 332 (2007) 1467-1475.

[14] **D. Ili´c, V. Rakoˇcevi´c,** Quasi contraction on a cone metric space, Applied Mathematics Letters 22 (2009) 728-731.

[15] **S. Jankovi´c, Z. Kadelburg, S. Radenovi´c**, On cone metric spaces, a survey, Nonlinear H. Rahimi, et al /IJIM Vol. 6, No. 2 (2014) 97-105 105

Analysis, Theory, Methods & Applications 74 (2011) 2591-2601.

[16] **G. Jungck,** Common fixed points for noncontinuous nonself maps on nonmetric spaces, Far East Journal of Mathematical Sciences 4 (1996) 199-215.

[17] **G. Jungck**, Commuting maps and fixed points, American Mathematic Monthly 83 (1976) 261-263.

[18] **O. Kada, T. Suzuki, W. Takahashi**, Nonconvex minimization theorems and fixed point theorems in complete metric spaces, Mathematica Japonica 44 (1996) 381-391.

[19] **R. Kannan**, Some results on fixed points, Bulletin of Calcutta Mathematical Society 60 (1968) 71-76.

[20] **H. Lakzian, F. Arabyani**, Some fixed point theorems in cone metric spaces with wdistance, International Journal of Mathematical Analysis 3 (2009) 1081-1086.

[21] **S. K. Mohanta**, Generalized w-distance and a fixed point theorem, International Journal of Contemporary Mathematical Sciences 6 (2011) 853-860.

[22] **H. Rahimi , G. Soleimani Rad ,** Fixed point theorems under c-distance in ordered cone metric space, Int. J. Industrial Mathematics (ISSN 2008-5621),Vol. 6, No. 2, 2014 Article ID IJIM-00253, 9 pages

[23] **A. Razani, Z. M. Nezhad, M. Boujary**, A fixed point theorem for w-distance, Applied Science 11 (2009) 114-117.

[24] **S. Rezapour, R. Hamlbarani**, Some note on the paper cone metric spaces and fixed point theorems of contractive mappings, Journal of Mathematical Analysis and Applications 345 (2008) 719-724.

[25] **B. E. Rhoades**, A comparison of various definition of contractive mappings, Transactions of the American Mathematical Society 266 (1977) 257-290.

[26] **R. Saadati, D. ORegan, S. M. Vaezpour, J. K. Kim,** Generalized distance and common fixed point theorems in Menger probabilistic metric spaces, Bulletin of the Iranian Mathematical Society 35 (2009) 97-117.

[27] **W. Shatanawi**, Partially ordered cone metric spaces and coupled fixed point results, Computers and Mathematics with Applications 60 (2010) 2508-2515.

[28] **W. Sintunavarat, Y. J. Cho, P. Kumam**, Common fixed point theorems for c-distance in ordered cone metric spaces, Computers and Mathematics with Applications 62 (2011) 1969-1978.

[29] **S. Wang, B. Guo,** Distance in cone metric spaces and common fixed point theorems, Applied Mathematics Letters 24 (2011) 17351739.

CHAPTER 3

Fixed Point Theorems for a Pair of Self Maps

in Ordered Cone Metric Spaces Under Lattice Ordered c-Distance

1 Introduction

Huang and Zhang [13] introduced the concept of the cone metric space, replacing the set of real numbers by an ordered Banach space, and they showed some fixed point theorems of contractive type mappings on cone metric spaces. Then, several fixed and common fixed point results in cone metric spaces were introduced in [2, 3, 9, 15, 24] and the references contained therein. The Banach contraction principle is the most celebrated fixed point theorem [6]. Afterward, some various definitions of contractive mappings were introduced by other researchers and several fixed and common fixed point theorems were considered in [7, 10, 17, 19, 25]. Also, the existence of fixed and common fixed points in partially ordered cone metric spaces was studied in [4, 5, 28]. In 1996, Kada et al. [18] defined the concept of *w*-distance in complete metric space and proved some fixed point theorems in complete metric spaces (see [1, 20, 21, 23]). Also Saadati et al. [26] introduced a probabilistic version of the *w*-distance of Kada et al. in a Menger probabilistic metric space. Cho et al. [8], and Wang and Guo [30] defined a concept of the c-distance in a cone metric space, which is a cone version of the *w*-distance of Kada et al. and proved some fixed point theorems in ordered cone metric spaces. Sintunavarat et al. [29] generalized the Banach contraction theorem on c-distance of Cho et al. [8]. Also, Dordevi´c et al. in [12] proved some fixed point and common fixed point theorems under c-distance for contractive mappings in tvs-cone metric spaces. H. Rahimi, G. Soleimani Rad [22] extended the Banach contraction principle [6] and Chatterjea contraction theorem [7] on c-distance of Cho et al. [8], and proved some fixed point and common fixed point theorems in ordered cone metric spaces. Sastry et al. [27] extended the results of [22] . Also they introduced the notion of lattice ordered c-distance and proved some fixed point theorems under a lattice ordered c-distance in ordered cone metric spaces, for a single function.

In this paper we extend the results of H. Rahimi & G. Soleimani Rad [22] and Sastry et al. [27] on common fixed points for a pair of self maps.

2 Preliminaries

First let us start with some basic definitions

Definition 2.1: ([11, 13])

Let E be a real Banach space and 0 denote the zero element in E. A subset P of E is said to be a cone if

(a) P is closed, non-empty and P $\neq \{0\}$;

(b) a,b $\in$ R, a,b ≥ 0, $x,y \in$ P imply that $ax + by \in$ P; where R is the real number system

(c) if $x \in$ P and $-x \in$ P, then $x = 0$.

Given a cone P $\subset$ E, we define a partial ordering $\leq$ with respect to P by $x \leq y \Leftrightarrow y - x \in$ P. We shall write $x < y$ if $x \leq y$ and $x \neq y$. Also, we write $x \ll y$ if and only if $y - x \in$ int P (where int P is interior of P). If int P $\neq \emptyset$, the cone P is called solid. The cone P is called normal if there is a number k > 0 such that for all $x, y \in$ E, $0 \leq x \leq y \Rightarrow \|x\| \leq k\|y\|$. The least positive number satisfying the above is called the normal constant of P.

Definition 2.2: ([13])

Let X be a nonempty set and E be a real Banach space equipped with the partial ordering $\leq$ with respect to the cone P $\subset$ E. Suppose that the mapping d : X × X → E satisfies:

(d1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then, d is called a cone metric on X and (X,d) is called a cone metric space.

Definition 2.3: ([13])

Let (X,d) be a cone metric space, $\{x_n\}$ a sequence in X and $x \in$ X. (i) $\{x_n\}$ converges to x if for every c $\in$ E with $0 \ll c$ there exists $n_0 \in$ N such that $d(x_n, x) \ll c$ for all $n > n_0$, and we write $\lim_{n\to\infty} d(x_n, x) = 0$.

(ii) $\{x_n\}$ is called a Cauchy sequence if for every $c \in E$ with $0 \ll c$ there exists $n_0 \in N$ such that $d(x_n, x_m) \ll c$ for all $m,n > n_0$, and we write $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.

(iii) If every Cauchy sequence in X is convergent, then X is called a complete cone metric space.

Lemma 2.4: ([13, 24])

Let (X,d) be a cone metric space and P be a normal cone with normal constant k. Also, let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y \in X$. Then the following hold:

(c1) $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$.

(c2) If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then $x = y$.

(c3) If $\{x_n\}$ converges to x, then $\{x_n\}$ is a Cauchy sequence.

(c4) If $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$.

(c5) $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n,m \to \infty$.

Lemma 2.5: ([4, 14]) Let E be a real Banach space with a cone P in E. Then, for all $u,v,w,c \in E$, the following hold:

(p1) If $u \leq v$ and $v \ll w$, then $u \ll w$.

(p2) If $0 \leq u \ll c$ for each $c \in \operatorname{int} P$, then $u = 0$.

(p3) If $u \leq \lambda u$ where $u \in P$ and $0 < \lambda < 1$, then $u = 0$.

(p4) Let $x_n \to 0$ in E, $0 \leq x_n$ and $0 \ll c$. Then there exists positive integer n_0 such that $x_n \ll c$ for each $n > n_0$.

(p5) If $0 \leq u \leq v$ and k is a nonnegative real number, then $0 \leq ku \leq kv$.

(p6) If $0 \leq u_n \leq v_n$ for all $n \in N$ and $u_n \to u$, $v_n \to v$ as $n \to \infty$, then $0 \leq u \leq v$.

Definition 2.6: ([8, 30])

Let (X,d) be a cone metric space. A function $q : X\times X \to E$ is called a c-distance on X if the following are satisfied:

(q1) $0 \leq q(x, y)$ for all $x, y \in X$;

(q2) $q(x, z) \leq q(x, y) + q(y, z)$ for all $x, y, z \in X$;

(q3) for all $n \geq 1$ and $x \in X$, if $q(x, y_n) \leq u$ for some u , then $q(x, y) \leq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$;

(q4) for all $c \in E$ with $0 \ll c$, there exists $e \in E$ with $0 \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

Remark 2.7: ([8, 30]) Each *w*-distance q in a metric space (X,d) is a c-distance (with $E = R^+$ and $P = [0,\infty)$). But the converse does not hold. Therefore, the c-distance is a generalization of *w*-distance.

Examples 2.8: ([8, 29, 30])

(1) Let (X,d) be a cone metric space and P be a normal cone. Put $q(x, y) = d(x, y)$ for all $x, y \in X$. Then q is a c-distance.

(2) Let $E = R$, $X = [0,\infty)$ and $P = \{x \in E : x \geq 0\}$. Define a mapping $d : X \times X \to E$ by $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a cone metric space. Define a mapping $q : X \times X \to E$ by $q(x, y) = y$ for all $x, y \in X$. Then q is a c distance.

(3) Let $E = C^1_{\ R}[0,1]$ with the norm $\|x\|= \|x\|_\infty+\|x'\|_\infty$ and consider the cone $P = \{x \in E : x(t) \geq 0 \text{ on } [0,1]\}$. Also, let $X = [0,\infty)$ and define a mapping

$d : X\times X \to E$ by $d(x, y) = |x-y|\psi$ for all $x, y \in X$, where $\psi : [0,1] \to R$ such that

$\psi(t) = 2t$. Then (X,d) is a cone metric space. Define a mapping $q : X \times X \to E$ by

$q(x, y) = (x + y)\psi$ for all $x, y \in X$. Then q is c-distance.

(4) Let (X,d) be a cone metric space and P be a normal cone. Put $q(x, y) = d(w, y)$

for all $x, y \in X$, where $w \in X$ is a fixed point. Then q is a c-distance.

Remark 2.9: ([8, 29, 30]) From Examples 2.8 (1,2,4), we have three important

results

(i) Each cone metric d on X with a normal cone is a c-distance q on X.

(ii) For c-distance q, $q(x, y) = 0$ is not necessarily equivalent to $x = y$ for all $x,y \in X$.

(iii) For c-distance q, $q(x,y) = q(y,x)$ does not necessarily hold for all $x,y \in X$.

Lemma 2.10: ([8, 29, 30])

Let (X,d) be a cone metric space and let q be a c-distance on X. Also, let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y, z \in X$. Suppose that $\{u_n\}$ and $\{v_n\}$ are two sequences in P converging to 0. Then the following hold:

(qp1) If $q(x_n, y) \leq u_n$ and $q(x_n, z) \leq v_n$ for $n \in N$, then $y = z$.

Specifically, if $q(x, y) = 0$ and $q(x, z) = 0$, then $y = z$.

(qp2) If $q(x_n, y_n) \leq u_n$ and $q(x_n, z) \leq v_n$ for $n \in N$, then $\{y_n\}$ converges to z.

(qp3) If $q(x_n, x_m) \leq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.

(qp4) If $q(y, x_n) \leq u_n$ for $n \in N$, then $\{x_n\}$ is a Cauchy sequence in X.

We use the following special case of (qp3) in Section 3

Lemma 2.11: [27]

In addition to the hypothesis of Lemma 2.10, assume the following :

Let $z_0 \in P$, $0 \leq \mu < 1$ and $q (x_n , x_m) \leq \mu^n z_0 \ \forall \ m \geq n$.

Then $\{ x_n \}$ is a Cauchy sequence. ($\because u_n = \mu^n z_0 \longrightarrow 0$ as $n \longrightarrow \infty$)

Definition 2.12: ([4, 8])

Let (X, ≤) be a partially ordered set. Two mappings $f, g : X \to X$ are said to be weakly increasing if $f x \leq g f x$ and $g x \leq f g x$ hold for all $x \in X$.

Definition 2.13: ([9])

A lattice is a partially ordered set S in which any two elements $a,b \in S$ have the supremum (a ∪ b) and the infimum (a ∩ b). Sometimes we write max { a,b } for (a ∪ b) and min { a,b } for (a ∩ b).

Definition 2.14: ([27)

Suppose (X,d) is a cone metric space and q : X × X → E is a c-distance. Clearly, the image q (X × X) of X × X under q is a subset of P. If the image
q (X × X) is a lattice in P, we say that q is a lattice ordered c-distance on X.

H. Rahimi, G. Soleimani Rad [22] proved following theorem.

Theorem 2.15 : ([22], Theorem 3.1)

Let (X, ≤) be a partially ordered set and (X,d) be a complete cone metric space. Also, let q be a c-distance on X and f : X → X be a continuous and non decreasing mapping with respect to ≤. Suppose that there exist mappings α,β,γ : X → [0,1) such that the following four conditions hold:

(i) $\alpha(fx) \le \alpha(x)$, $\beta(fx) \le \beta(x)$ and $\gamma(fx) \le \gamma(x)$ for all $x \in X$;

(ii) $(\alpha + 2\beta + 2\gamma)(x) < 1$ for all $x \in X$;

(iii) for all $x,y \in X$ with $x \le y$, $q(fx,fy) \le \alpha(x)q(x,y) + \beta(x)q(x,fy) + \gamma(x)q(y,fx)$;

(iv) for all $x,y \in X$ with $x \le y$, $q(fy,fx) \le \alpha(x)q(y,x) + \beta(x)q(fy,x) + \gamma(x)q(fx,y)$.

If there exists $x_0 \in X$ such that $x_0 \le fx_0$, then f has a fixed point. Moreover, if

$fz = z$, then $q(z,z) = 0$.

Note : In the above theorem α, β, γ are functions of x. In section 3, we take α, β, γ to be constants so that above (i) becomes obvious.

By introducing lattice order on the image of q, the following theorem is proved in [27].

Theorem 2.16 : (Sastry etal. [27], Theorem 3.3)

Let (X , ≤) be a partially ordered set and (X, d) be a complete cone metric space. Also, let q be a lattice ordered c-distance on X and f : X → X be continuous non decreasing mapping with respect to ≤ . Suppose $\lambda \in [\,0, \tfrac{1}{2}\,)$ and q satisfies the following conditions :

(i) $x \le y \le z$ implies

$$q(x,y) \le q(x,z)\ ;\ q(y,x) \le q(z,x) \qquad \ldots(3.3.1)$$

$$q(y,z) \le q(x,z)\ ;\ q(z,y) \le q(z,x) \quad \forall\ x,y,z \in X \qquad \ldots(3.3.2)$$

(ii) $q(fx,fy) \le \lambda\, max\{q(x,y), q(x,fy), q(y,fx)\}$ $if\ x \le y$ … (3.3.3)

(iii) $q(fy,fx) \le \lambda\, max\{q(y,x), q(fy,x), q(fx,y)\}$ $if\ x \le y$ ...(3.3.4)

Suppose there exists $x_0 \in X$ such that $x_0 \le f x_0$. Then f has a fixed point in X and no two fixed points are comparable. Moreover, if $f z = z$, then $q(z,z) = 0$.

The following theorem is an extension of Theorem 2.15([22], Theorem 3.1) to a

pair of functions.

Theorem 2.17 : (Theorem 3.3 , [22])

Let (X, ≤) be a partially ordered set and (X,d) be a complete cone metric space. Also, let q be a c-distance on X and f, g : X → X be two continuous and weakly increasing mappings with respect to ≤. Suppose that there exist mappings α,β,γ : X → [0,1) such that the following five conditions hold :

(t 1) $\alpha(fx) \le \alpha(x)$, $\beta(fx) \le \beta(x)$ and $\gamma(fx) \le \gamma(x)$ for all $x \in X$

(t 2) $\alpha(gx) \le \alpha(x)$, $\beta(gx) \le \beta(x)$ and $\gamma(gx) \le \gamma(x)$ for all $x \in X$

(t 3) $(\alpha + 2\beta + 2\gamma)(x) < 1$ for all $x \in X$;

(t 4) for all comparable $x, y \in X$,

$$q(fx, gy) \le \alpha(x)\, q(x,y) + \beta(x)\, q(x,gy) + \gamma(x)\, q(y,fx)$$

(t 5) for all comparable x,y ∈ X,

$$q(gy, fx) \le \alpha(x)\, q(y,x) + \beta(x)\, q(gy,x) + \gamma(x)\, q(fx,y)$$

Then f and g have a common fixed point. Moreover, if $fz = gz = z$,

then $q(z,z) = 0$.

Note : In this theorem α , β , γ are functions of x. In section 3, we take α , β , γ to be constants so that (t_1) and (t_2) become obvious.

3 Main results

In this section , we extend the results of H. Rahimi, G. Soleimani [22] and Sastry et al [27], taking q to be a lattice ordered c-distance. We also establish some more common fixed point theorems under lattice ordered c-distance in ordered cone metric spaces, for a pair of self maps.

First we are proving the following Lemma :

Lemma 3.1: If $q (x_n , x_{n+1}) \leq \mu \, q (x_{n-1} , x_n) \ \forall n$ where $0 \leq \mu < 1$ then $\{x_n\}$ is a Cauchy sequence in X.

Proof : We observe that $q (x_n , x_{n+1}) \leq \mu^n \, q (x_0 , x_1)$ for n = 1,2,…

Now let m > n,

$$q (x_n , x_m) \leq q (x_n , x_{n+1}) + q (x_{n+1} , x_{n+2}) + \ldots + q (x_{m-1} , x_m)$$

$$\leq (\mu^n + \mu^{n+1} + \ldots + \mu^{m-1}) \, q (x_0 , x_1)$$

$$\leq \mu^n \left(\frac{1}{1-\mu}\right) q (x_0 , x_1)$$

$$\leq \mu^n z_0 \text{ where } z_0 = \left(\frac{1}{1-\mu}\right) q (x_0 , x_1)$$

Hence by Lemma 2.11, $\{x_n\}$ is a Cauchy sequence in X.

Now we state and prove two of our main results with the underlying space X having a lattice ordered c-distance in ordered cone metric spaces, for a pair of self maps. This becomes necessary since we consider maximum of three terms in the control function which we cannot do if the c-distance is not lattice ordered.

Theorem 3.2 : Let $(X , \leq)$ be a partially ordered set and (X, d) be a complete cone metric space. Also, let q be a lattice ordered c-distance on X and $f, g : X \to X$ be two continuous and weakly increasing mappings with respect to $\leq$. Suppose $\lambda \in [0, ½)$ and q satisfies the following conditions :

(i) $x \leq y \leq z$ implies

$$q(x,y) \leq q(x,z) \; ; \; q(y,x) \leq q(z,x) \qquad \ldots(3.2.1)$$

$$q(y,z) \leq q(x,z) \; ; \; q(z,y) \leq q(z,x) \;\; \forall \;\; x,y,z \in X \qquad \ldots(3.2.2)$$

(ii) $q(fx,gy) \leq \lambda \max\{q(x,y), q(x,gy), q(y,fx)\} \;\; if \; x \leq y \qquad \ldots(3.2.3)$

(iii) $q(gx,fy) \leq \lambda \max\{q(x,y), q(x,fy), q(y,gx)\} \;\; if \; x \leq y \qquad \ldots(3.2.4)$

Suppose there exists $x_0 \in X$ such that $x_0 \leq f x_0$ … (3.2.5)

Then f and g have a common fixed point in X. Moreover, if $f z = gz = z$, then

$q(z,z) = 0$.

Proof : Let x_0 be as in (3.2.5). We construct the sequence $\{x_n\}$ in X as follows:

$x_{2n+1} = fx_{2n}$, $x_{2n+2} = gx_{2n+1}$ $\forall$ n = 0,1,2,…

Since f and g are weakly increasing mappings, such that

$x_1 = f x_0 \leq g f x_0 = g x_1 = x_2$, $x_2 = g x_1 \leq f g x_1 = f x_2 = x_3$.

If we continue in this manner, then there exist $x_{2n+1} \in X$

$x_{2n+1} = f x_{2n} \leq g f x_{2n} = g x_{2n+1} = x_{2n+2}$ and $x_{2n+2} \in X$

$x_{2n+2} = g x_{2n+1} \leq f g x_{2n+1} = f x_{2n+2} = x_{2n+3}$ for n = 0,1,2,…

Thus, $x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots$ for all $n \geq 1$.

That is $\{x_n\}$ is a non decreasing sequence. Since $x_{2n} \leq x_{2n+1}$ for all $n \geq 1$ and by using (3.2.3)

for $x = x_{2n}$ and $y = x_{2n+1}$, we have

$q(x_{2n+1}, x_{2n+2}) = q(f x_{2n} , g x_{2n+1})$

$\leq \lambda \max \{ q(x_{2n}, x_{2n+1}), q(x_{2n}, gx_{2n+1}), q(x_{2n+1}, fx_{2n})\}$

$= \lambda \max \{ q(x_{2n}, x_{2n+1}), q(x_{2n}, x_{2n+2}), q(x_{2n+1}, x_{2n+1})\}$

$= \lambda\ q\ (x_{2n}, x_{2n+2})$ (from 3.2.1, $x_{2n} \leq x_{2n+1} \leq x_{2n+2}$)

$\leq \lambda \{ q\ (x_{2n}, x_{2n+1}) + q\ (x_{2n+1}, x_{2n+2}) \}$

$\Longrightarrow q\ (x_{2n+1}, x_{2n+2}) \leq (\frac{\lambda}{1-\lambda})\ q\ (x_{2n}, x_{2n+1})$ …(3.2.6)

Similarly, using (3.2.4), we get

$q\ (x_{2n}, x_{2n+1})$

$= q\ (gx_{2n-1}, fx_{2n})$

$\leq \lambda\ max \{ q\ (x_{2n-1}, x_{2n}), q\ (x_{2n-1}, fx_{2n}), q\ (x_{2n}, gx_{2n-1})\}$ (∵ by (3.2.4))

$= \lambda\ max \{ q\ (x_{2n-1}, x_{2n}), q\ (x_{2n-1}, x_{2n+1}), q\ (x_{2n}, x_{2n})\}$

$= \lambda\ q\ (x_{2n-1}, x_{2n+1})$ (from (3.2.2))

$\leq \lambda \{ q\ (x_{2n-1}, x_{2n}) + q\ (x_{2n}, x_{2n+1}) \}$

$\Longrightarrow q\ (x_{2n}, x_{2n+1}) \leq (\frac{\lambda}{1-\lambda})\ q\ (x_{2n-1}, x_{2n})$ …(3.2.7)

From (3.2.6) and (3.2.7) we get

$q\ (x_{n+1}, x_{n+2}) \leq (\frac{\lambda}{1-\lambda})\ q\ (x_n, x_{n+1})$

∴ By induction ,

$q\ (x_{n+1}, x_{n+2}) \leq (\frac{\lambda}{1-\lambda})^{n+1}\ q\ (x_0, x_1)$, n = 0,1,2,…

Now $\frac{\lambda}{1-\lambda} < 1$ *since* $0 \leq \lambda < ½$

∴ By taking $\mu = \frac{\lambda}{1-\lambda}$ and $z_0 = q\ (x_0, x_1) \in P$, from Lemma 3.1, we get that

$\{ x_n \}$ is a Cauchy sequence in X. Since X is complete, there exists

a point z ∈ X such that $x_n \rightarrow$ z as n → ∞ .

∴ By continuity of f and g we get

$x_{n+1} = f\ x_n \rightarrow f z$ and $x_{n+2} = g\ x_{n+1} \rightarrow g\ z$ as n → ∞ .

Hence $fz = z$ and $gz = z$. Thus, z is a fixed point of f and g.

Then $\quad q\,(z,z) = q\,(fz,gz)$

$$\leq \lambda\, max\,\{\, q\,(z,z), q\,(z,gz), q\,(z,fz)\} \quad (\text{by } (3.2.3))$$

$$= \lambda\, max\,\{\, q\,(z,z), q\,(z,z), q\,(z,z)\}$$

$$= \lambda\, q\,(z,z)$$

$$\Rightarrow q\,(z,z) = 0$$

The following theorem which is an analogue of theorem 3.2, for decreasing functions can be easily established.

Theorem 3.3 : Let $(X, \leq)$ be a partially ordered set and (X, d) be a complete cone metric space. Also, let q be a lattice ordered c-distance on X and f, g : X → X be two continuous and weakly decreasing mappings with respect to ≤ . Suppose $\lambda \in [\,0, ½\,)$ and q satisfies the following conditions :

(i) $x \leq y \leq z$ implies

(a) $q(x,y) \leq q(x,z)$; $q(y,x) \leq q(z,x)$ …(3.3.1)

(b) $q(y,z) \leq q(x,z)$; $q(z,y) \leq q(z,x)$ $\forall\ x,y,z \in X$ … (3.3.2)

(ii) $q(fx,gy) \leq \lambda \max\{q(x,y), q(x,gy), q(fx,y)\}$ $if\ x \leq y$ … (3.3.3)

(iii) $q(gx,fy) \leq \lambda \max\{q(x,y), q(x,fy), q(gx,y)\}$ $if\ x \leq y$ ... (3.3.4)

Suppose there exists $x_0 \in X$ such that $x_0 \geq f x_0$ … (3.3.5)

Then f and g have a common fixed point in X. Moreover, if $fz = gz = z$, then

q(z,z) = 0.

Now we prove an improved version of theorem 2.17

Theorem 3.4 : Let (X,≤) be a partially ordered set and (X,d) be a complete cone metric space. Also, let q be a c-distance on X and f , g : X → X be a continuous and non decreasing mappings with respect to ≤. Suppose there exists $x_0 \in X$ such that $x_0 \leq f x_0$.

Define a sequence $\{x_n\}$ by $x_{2n+1} = fx_{2n}$, $x_{2n+2} = gx_{2n+1}$ $\forall$ n = 0,1,2,… …(3.4.1)

Suppose there exist α, β, γ with $\alpha + 2\beta + 2\gamma < 1$ (non negative constants)

such that

$$q(x_{2n+1}, x_{2m+2}) \leq \alpha q(x_{2n}, x_{2m+1}) + \beta q(x_{2n}, x_{2m+2}) + \gamma q(x_{2m+1}, x_{2n+1}) \ \forall \text{ m,n} \quad \dots(3.4.2)$$

$$q(x_{2m+2}, x_{2n+1}) \leq \alpha q(x_{2m+1}, x_{2n}) + \beta q(x_{2m+2}, x_{2n}) + \gamma q(x_{2n+1}, x_{2m+1}) \ \forall \text{ m,n} \quad \dots(3.4.3)$$

Then $\{x_n\}$ is a Cauchy sequence with limit z (say) and z is a common fixed point of f and g. Moreover, if $fz = z$, then $q(z,z) = 0$.

Proof : Since f and g are weakly increasing mappings, such that

$x_1 = f\,x_0 \leq g\,f\,x_0 = g\,x_1 = x_2$, $x_2 = g\,x_1 \leq f\,g\,x_1 = f\,x_2 = x_3$.

If we continue in this manner, then there exist $x_{2n+1} \in X$

$x_{2n+1} = f\,x_{2n} \leq g\,f\,x_{2n} = g\,x_{2n+1} = x_{2n+2}$ and $x_{2n+2} \in X$

$x_{2n+2} = g\,x_{2n+1} \leq f\,g\,x_{2n+1} = f\,x_{2n+2} = x_{2n+3}$ for n = 0,1,2,…

Thus, $x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots$ for all $n \geq 1$.

That is $\{x_n\}$ is a non decreasing sequence. Since $x_{2n} \leq x_{2n+1}$ for all $n \geq 1$ and by

Put m = n-1 in (3.4.2) & (3.4.3) we get

$$q(x_{2n+1}, x_{2n})$$

$$\leq \alpha q(x_{2n}, x_{2n-1}) + \beta q(x_{2n}, x_{2n}) + \gamma q(x_{2n-1}, x_{2n+1})$$

$$\leq \alpha q(x_{2n}, x_{2n-1}) + \beta\{q(x_{2n}, x_{2n-1}) + q(x_{2n-1}, x_{2n})\} + \gamma\{q(x_{2n-1}, x_{2n}) + q(x_{2n}, x_{2n+1})\}$$

$$\therefore q(x_{2n+1}, x_{2n}) \leq \alpha q(x_{2n}, x_{2n-1}) + \beta\{q(x_{2n}, x_{2n-1}) + q(x_{2n-1}, x_{2n})\} + \gamma\{q(x_{2n-1}, x_{2n}) + q(x_{2n}, x_{2n+1})\} \quad \dots(3.4.4)$$

Similarly from (3.4.3) we get

$$q(x_{2n}, x_{2n+1}) \leq \alpha q(x_{2n-1}, x_{2n}) + \beta\{q(x_{2n}, x_{2n-1}) + q(x_{2n-1}, x_{2n})\} +$$
$$\gamma\{q(x_{2n+1}, x_{2n}) + q(x_{2n}, x_{2n-1})\} \qquad \ldots(3.4.5)$$

Combining (3.4.4) and (3.4.5) we get

$$q(x_{2n+1}, x_{2n}) + q(x_{2n+2}, x_{2n+1})$$
$$\leq (\alpha + 2\beta + \gamma)\{q(x_{2n}, x_{2n-1}) + q(x_{2n-1}, x_{2n})\} +$$
$$\gamma\{ q(x_{2n+1}, x_{2n}) + q(x_{2n}, x_{2n+1})\}$$

$\Longrightarrow \lambda_{2n} \leq (\alpha + 2\beta + \gamma)\, \lambda_{2n-1} + \gamma\, \lambda_{2n}$ where

$\lambda_n = q(x_n, x_{n+1}) + q(x_{n+1}, x_n) \Longrightarrow \lambda_{2n} = q(x_{2n}, x_{2n+1}) + q(x_{2n+1}, x_{2n})$

$$\Longrightarrow \quad \lambda_{2n} \leq \left(\frac{\alpha+2\beta+\gamma}{1-\gamma}\right) \lambda_{2n-1}$$

$$\therefore \quad \lambda_{2n} \leq \mu\, \lambda_{2n-1}, \; n = 1,2,3,\ldots \qquad \ldots(3.4.6)$$

where $\mu = \max\{ \frac{\alpha+2\beta+\gamma}{1-\gamma}, \frac{\alpha+\beta+2\gamma}{1-\beta} \} < 1$

Now put m = n in (3.4.2) & (3.4.3) we get

$$q(x_{2n+1}, x_{2n+2}) \leq \alpha q(x_{2n}, x_{2n+1}) + \beta q(x_{2n}, x_{2n+2}) + \gamma q(x_{2n+1}, x_{2n+1})$$
$$\leq \alpha q(x_{2n}, x_{2n+1}) + \beta \{ q(x_{2n}, x_{2n+1}) + q(x_{2n+1}, x_{2n+2}) \}$$
$$+ \gamma \{ q(x_{2n+1}, x_{2n}) + \gamma q(x_{2n}, x_{2n+1})$$

$$\therefore \; q(x_{2n+1}, x_{2n+2}) \leq \alpha q(x_{2n}, x_{2n+1}) + \beta \{ q(x_{2n}, x_{2n+1}) + q(x_{2n+1}, x_{2n+2}) \}$$
$$+ \gamma \{ q(x_{2n+1}, x_{2n}) + \gamma q(x_{2n}, x_{2n+1}) \qquad \ldots(3.4.7)$$

Similarly from (3.4.3) we get

$$q(x_{2n+2}, x_{2n+1}) \leq \alpha q(x_{2n+1}, x_{2n}) + \beta \{ q(x_{2n+1}, x_{2n}) + q(x_{2n+2}, x_{2n+1}) \}$$
$$+ \gamma \{ q(x_{2n}, x_{2n+1}) + \gamma q(x_{2n+1}, x_{2n}) \qquad \ldots(3.4.8)$$

Combining (3.4.7) and (3.4.8) we get

$q(x_{2n+1}, x_{2n+2}) + q(x_{2n+2}, x_{2n+1})$

$\leq (\alpha + \beta + 2\gamma)\{q(x_{2n}, x_{2n+1}) + q(x_{2n+1}, x_{2n})\} +$

$$\beta\{ q(x_{2n+1}, x_{2n+2}) + q(x_{2n+2}, x_{2n+1})\}$$

$\lambda_{2n+1} \leq (\alpha + \beta + 2\gamma)\, \lambda_{2n} + \beta\, \lambda_{2n+1}$ where

$\lambda_n = q(x_n, x_{n+1}) + q(x_{n+1}, x_n) \Longrightarrow \lambda_{2n+1} = q(x_{2n+1}, x_{2n+2}) + q(x_{2n+2}, x_{2n+1})$

$\Longrightarrow \quad \lambda_{2n+1} \leq \left(\frac{\alpha+\beta+2\gamma}{1-\beta}\right) \lambda_{2n}$

$\therefore \quad \lambda_{2n+1} \leq \mu\, \lambda_{2n}$, n = 0,1,2,3,… …(3.4.9)

where $\mu = \max\{ \frac{\alpha+2\beta+\gamma}{1-\gamma}, \frac{\alpha+\beta+2\gamma}{1-\beta} \} < 1$

From (3.4.6) & (3.4.9) we get $\lambda_{n+1} \leq \mu\, \lambda_n \ \forall n$ with $\mu < 1$

$\therefore \lambda_n$ is Cauchy.

Hence by Lemma 3.1, $\{ x_n \}$ is a Cauchy sequence in X. Since X is complete,

there exists a point $z \in X$ such that $x_n \rightarrow z$ as $n \rightarrow \infty$.

$\therefore$ By continuity of f we get

$x_{2n+1} = f\, x_{2n} \rightarrow f z$ and $x_{2n+2} = g\, x_{2n+1} \rightarrow g\, z$ as $n \rightarrow \infty$.

Hence $f z = z$ and $g z = z$. Thus, z is a fixed point of f and g.

Suppose that $z \in X$ is any point satisfying $f z = g z = z$

Then $\quad q(z,z) = q(fz, gz)$

$$\leq \alpha q(z,z) + \beta q(z, gz) + \gamma q(z, fz)$$

$$= \alpha q(z,z) + \beta q(z,z) + \gamma q(z,z)$$

$$= (\alpha + \beta + \gamma)\, q(z,z)$$

$$< q(z,z) \qquad (\because \alpha + \beta + \gamma < \alpha + 2\beta + 2\gamma < 1)$$

$$\Longrightarrow q(z,z) = 0$$

Now we show that theorem 2.17 is a simple consequence of theorem 3.4.

Corollary 3.5 : (H. Rahimi , G. Soleimani Rad [22], Theorem 3.3)

Let (X,≤) be a partially ordered set and (X,d) be a complete cone metric space. Also, let q be a c-distance on X and f, g : X → X be two continuous and weakly increasing mappings with respect to ≤. Suppose that there exist mappings
α,β,γ : X → [0,1) such that the following five conditions hold :

(t1) $\alpha(fx) \leq \alpha(x)$, $\beta(fx) \leq \beta(x)$ and $\gamma(fx) \leq \gamma(x)$ for all $x \in X$

(t2) $\alpha(gx) \leq \alpha(x)$, $\beta(gx) \leq \beta(x)$ and $\gamma(gx) \leq \gamma(x)$ for all $x \in X$

(t3) $(\alpha + 2\beta + 2\gamma)(x) < 1$ for all $x \in X$;

(t4) for all comparable $x, y \in X$,

$$q(fx, gy) \leq \alpha(x)\, q(x,y) + \beta(x)\, q(x, gy) + \gamma(x)\, q(y, fx) \qquad \text{...(3.5.1)}$$

(t5) for all comparable x,y ∈ X,

$$q(gy, fx) \leq \alpha(x)\, q(y,x) + \beta(x)\, q(gy, x) + \gamma(x)\, q(fx, y) \qquad \text{...(3.5.2)}$$

Suppose there exists $x_0 \in X$ such that $x_0 \leq fx_0$.

Then f and g have a common fixed point. Moreover, if $fz = gz = z$,

then $q(z,z) = 0$.

Proof. : Let $\alpha = \alpha(x_0)$, $\beta = \beta(x_0)$, $\gamma = \gamma(x_0)$

Then take $x = x_{2n}$, $y = x_{2m+1}$ in (3.5.1) we get

$q(x_{2n+1}, x_{2m+2})$

$= q(fx_{2n}, gx_{2m+1})$

$\leq \alpha(x_{2n})q(x_{2n}, x_{2m+1}) + \beta(x_{2n})q(x_{2n}, x_{2m+2}) + \gamma(x_{2n})q(x_{2m+1}, x_{2n+1})$

$\leq \alpha(x_0)q(x_{2n}, x_{2m+1}) + \beta(x_0)q(x_{2n}, x_{2m+2}) + \gamma(x_0)q(x_{2m+1}, x_{2n+1})$

($\because \alpha(x_{2n}) \leq \alpha(x_0)$ for n = 1,2,3,…)

$= \alpha\ q(x_{2n}, x_{2m+1}) + \beta\ q(x_{2n}, x_{2m+2}) + \gamma\ q(x_{2m+1}, x_{2n+1})$

where $\alpha = \alpha(x_0)$, $\beta = \beta(x_0)$ and $\gamma = \gamma(x_0)$

$\therefore$ $q(x_{2n+1}, x_{2m+2}) \leq$

$$\alpha q(x_{2n}, x_{2m+1}) + \beta q(x_{2n}, x_{2m+2}) + \gamma q(x_{2m+1}, x_{2n+1}) \quad \ldots(3.5.3)$$

Similarly from (3.5.2) we get

$q(x_{2m+2}, x_{2n+1}) \leq$

$$\alpha q(x_{2m+1}, x_{2n}) + \beta q(x_{2m+2}, x_{2n}) + \gamma q(x_{2n+1}, x_{2m+1}) \quad \ldots(3.5.4)$$

Combining (3.5.3) and (3.5.4) follows that the hypothesis of Theorem 3.4 is satisfied. Hence from theorem 3.4 the result follows.

Acknowledgement :

The third author gratefully acknowledges **(i)** the financial support from UGC under UGC Minor Research Project, NO.F: MRP – 4536/14 (SERO / UGC) in carrying on this research **(ii)** the Principal Dr. Sr. Shyji and the management of St.Joseph's College for Women(A),Visakhapatnam for providing the necessary permission **(iii)** the authorities of GITAM for providing necessary facilities to carry on this research.

References

[1] **M. Abbas, D. Ili´c, M. Ali Khan**, Coupled coincidence point and coupled common fixed point theorems in partially ordered metric spaces with w-distance, Fixed Point Theory and Applications 11 (2010) Article ID 134897.

[2] **M. Abbas, G. Jungck**, Common fixed point results for non-commuting mappings without continuity in cone metric spaces, Journal of Mathematical Analysis and Applications 341 (2008) 416-420.

[3] **M. Abbas, B. E. Rhoades, T. Nazir**, Common fixed points for four maps in cone metric spaces, Applied Mathematics and Computation 216 (2010) 80-86.

[4] **I. Altun, B. Damnjanovi´c, D. Djori´c,** Fixed point and common fixed point theorems on ordered cone metric spaces, Applied Mathematics Letters 23 (2010) 310-316.

[5] **I. Altun, G. Durmaz**, Some fixed point theorems on ordered cone metric spaces, Rendiconti del Circolo Matematico di Palermo 58 (2009) 319-325.

[6] **S. Banach**, Sur les oprations dans les ensembles abstraits et leur application aux equations intgrales, Fundamenta Mathematicae 3 (1922) 133-181.

[7] **S. K. Chatterjea**, Fixed point theorems, Comptes rendus de l'Academie bulgare des Sciences 25 (1972) 727-730.

[8] **Y. J. Cho, R. Saadati, S. H. Wang**, Common fixed point theorems on generalized distance in ordered cone metric spaces, Computers and Mathematics with Applications 61 (2011) 1254-1260.

[9] **B. S. Choudhury, N. Metiya**, The point of coincidence and common fixed point for a pair of mappings in cone metric spaces, Computers and Mathematics with Applications 60 (2010) 1686-1695.

[10] **L. B. ´Ciri´c,** A generalization of Banach's contraction principle, Proceedings of the American Mathematical Society 45 (1974) 267-273.

[11] **K. Deimling**, Nonlinear Functional Analysis, Springer-Verlag, (1985).

[12] **M. Dordevi´c, D. Dori´c, Z. Kadelburg, S. Radenovi´c, D. Spasi´c,** Fixed point results under c-distance in tvs-cone metric spaces, Fixed Point Theory and Applications (2011) http://dx.doi.org/10.1186/ 1687-1812-2011-29/.

[13] **L. G. Huang, X. Zhang,** Cone metric spaces and fixed point theorems of contractive mappings, Journal of Mathematical Analysis and Applications 332 (2007) 1467-1475.

[14] **D. Ili´c, V. Rakoˇcevi´c,** Quasi contraction on a cone metric space, Applied Mathematics Letters 22 (2009) 728-731.

[15] **S. Jankovi´c, Z. Kadelburg, S. Radenovi´c**, On cone metric spaces, a survey, Nonlinear H. Rahimi, et al /IJIM Vol. 6, No. 2 (2014) 97-105 105

Analysis, Theory, Methods & Applications 74 (2011) 2591-2601.

[16] **G. Jungck,** Common fixed points for noncontinuous nonself maps on nonmetric spaces, Far East Journal of Mathematical Sciences 4 (1996) 199-215.

[17] **G. Jungck**, Commuting maps and fixed points, American Mathematic Monthly 83 (1976) 261-263.

[18] **O. Kada, T. Suzuki, W. Takahashi**, Nonconvex minimization theorems and fixed point theorems in complete metric spaces, Mathematica Japonica 44 (1996) 381-391.

[19] **R. Kannan**, Some results on fixed points, Bulletin of Calcutta Mathematical Society 60 (1968) 71-76.

[20] **H. Lakzian, F. Arabyani**, Some fixed point theorems in cone metric spaces with wdistance, International Journal of Mathematical Analysis 3 (2009) 1081-1086.

[21] **S. K. Mohanta**, Generalized w-distance and a fixed point theorem, International Journal of Contemporary Mathematical Sciences 6 (2011) 853-860.

[22] **H. Rahimi , G. Soleimani Rad ,** Fixed point theorems under c-distance in ordered cone metric space, Int. J. Industrial Mathematics (ISSN 2008-5621),Vol. 6, No. 2, 2014 Article ID IJIM-00253, 9 pages

[23] **A. Razani, Z. M. Nezhad, M. Boujary**, A fixed point theorem for w-distance, Applied Science 11 (2009) 114-117.

[24] **S. Rezapour, R. Hamlbarani**, Some note on the paper cone metric spaces and fixed point theorems of contractive mappings, Journal of Mathematical Analysis and Applications 345 (2008) 719-724.

[25] **B. E. Rhoades**, A comparison of various definition of contractive mappings, Transactions of the American Mathematical Society 266 (1977) 257-290.

[26] **R. Saadati, D. ORegan, S. M. Vaezpour, J. K. Kim,** Generalized distance and common fixed point theorems in Menger probabilistic metric spaces, Bulletin of the Iranian Mathematical Society 35 (2009) 97-117.

[27] **K.P.R. Sastry, A.Chandra Sekhar, K.Sujatha**, Fixed Point Theorems for a Self Map in Ordered Cone Metric Spaces Under c-Distance, communicated

[28] **W. Shatanawi**, Partially ordered cone metric spaces and coupled fixed point results, Computers and Mathematics with Applications 60 (2010) 2508-2515.

[29] **W. Sintunavarat, Y. J. Cho, P. Kumam**, Common fixed point theorems for c-distance in ordered cone metric spaces, Computers and Mathematics with Applications 62 (2011) 1969-1978.

[30] **S. Wang, B. Guo,** Distance in cone metric spaces and common fixed point theorems, Applied Mathematics Letters 24 (2011) 17351739.